Arithmetic-geometric mean inequality

The arithmetic mean of *n* numbers, $a_1,\ldots,a_n$, is defined to be $\dfrac{a_1+\cdots+a_n}{n}$.

The geometric mean of *n* positive numbers, $a_1,\ldots,a_n$, is defined to be $\sqrt[n]{a_1\ldots a_n}$.

The Arithmetic-geometric mean inequality states that '*the geometric mean of a set of positive numbers is less than or equal to the arithmetic mean of the same set of numbers*' *ie*:

$$\sqrt[n]{a_1\ldots a_n} \le \frac{a_1+\cdots+a_n}{n}$$

Example

Show that the arithmetic-geometric mean inequality holds for the numbers 7 and 9.

Solution

Arithmetic mean $= \dfrac{7+9}{2} = 8$

Geometric mean $= \sqrt{7\times 9} = \sqrt{63} = 7.94$

So it does hold for these two numbers.

Question 1.1

When does the arithmetic mean of two numbers equal the geometric mean?

Solutions

Solution 1.1

The arithmetic and geometric means are equal when the two numbers are equal. This can be proved as follows:

$$\frac{a+b}{2}=\sqrt{ab} \Rightarrow \frac{(a+b)^2}{4}=ab \Rightarrow (a+b)^2=4ab$$

This simplifies to $(a-b)^2=0$, ie $a=b$.