Complex numbers

Basic algebra

A complex number is represented in the form $a+ib$, where $i=\sqrt{-1}$, and *a* and *b* are real numbers. a is called the 'real part' and *b* is called the 'imaginary part'. You may also see complex numbers written with *j*'s, rather than *i*'s.

Complex numbers can be added, subtracted and multiplied as follows:

(i) $(a+ib)+(c+id)=(a+c)+i(b+d)$

(ii) $(a+ib)-(c+id)=(a-c)+i(b-d)$

(iii) $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

The rules for addition and subtraction just require the real and imaginary parts to be added separately. The logic of the rule for multiplication can be seen by multiplying out the brackets and using the fact that $i^2=-1$.

Question 1.1

Find:

(i) $(2+3i)+(3-4i)$ (ii) $(4-6i)-(3i-7)$

(iii) $(3-4i)(6+2i)$ (iv) $(4+2i)(4-2i)$

The answer to last part of this question, where the product of two complex numbers is a real number, leads to an important property of complex numbers.

$a+ib$ and $a-ib$ are called 'complex conjugates'. Their product is the real number a^2+b^2. The complex conjugate of the number $a+ib$ is $a-ib$, and vice versa.

This enables us to divide complex numbers.

Example

Simplify $\frac{2-3i}{1+2i}$.

Solution

If we multiply the numerator and denominator by the complex conjugate of the denominator, we get:

$$\frac{2-3i}{1+2i}\times\frac{1-2i}{1-2i}=\frac{2-3i-4i-6}{1+4}=\frac{-4-7i}{5}=\frac{1}{5}(-4-7i)$$

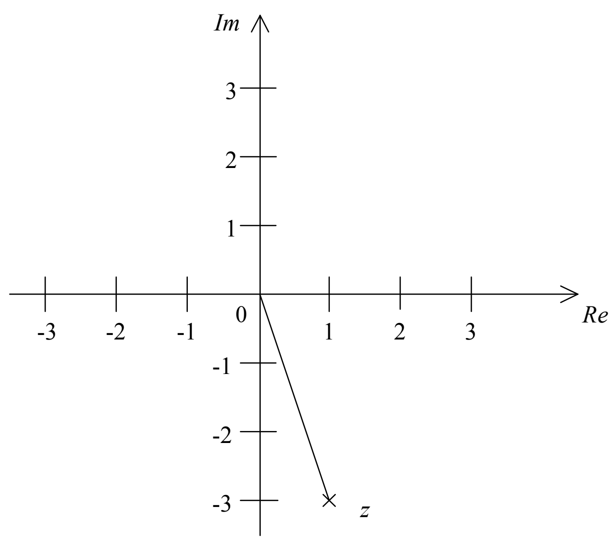
This is in the form $a+ib$, where $a=-0.8$ and $b=-1.4$.

Question 1.2

Simplify $\frac{3i-4}{1-4i}$.

Argand diagrams

Complex numbers can be represented on an 'Argand diagram', where you plot the imaginary part against the real part. For example, the complex number $z=1-3i$ can be shown as follows:



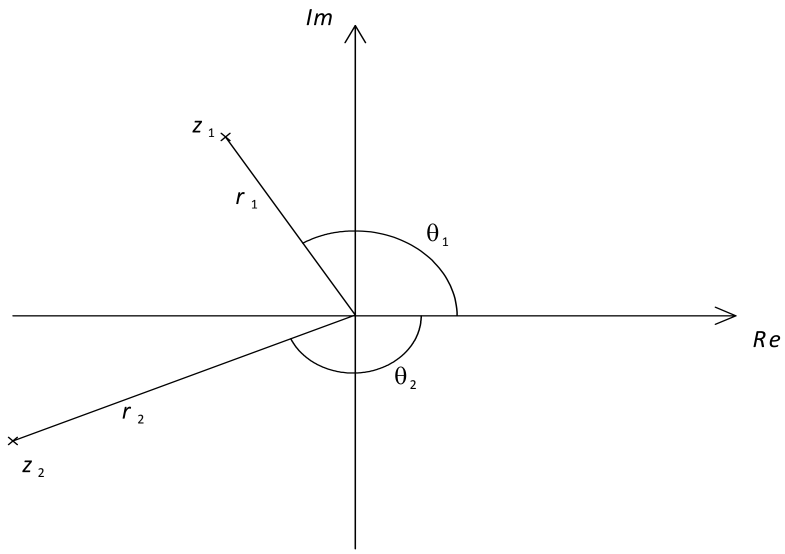
Notice that it is the *point* that represents the complex number, not the line. There is a reason for including the line though, which we will describe shortly.

By representing a complex number on an Argand diagram, it is easy to picture two further properties of complex numbers.

The modulus, *r*, of a complex number $a+ib$ is given by $r=\sqrt{a^2+b^2}$, *ie* the square root of the sum of the real part squared and the imaginary part squared. It is equivalent to the length of the line shown on the Argand diagram.

The argument, θ , of a complex number $a+ib$ is defined to be the angle between the line and the positive *x*-axis on the Argand diagram. The angle is measured in radians rather than degrees and can take values $-\pi<\theta\leq\pi$. You'll need to ensure that your calculator is set to radian mode when working with the argument in trigonometric functions. A formal definition of radians is given in the Appendix for interested students. In summary π radians is equivalent to 180º.

This diagram illustrates the modulus and argument for two complex numbers z_1 and z_2 , with moduli r_1 and r_2 , and arguments θ_1 and θ_2 respectively.

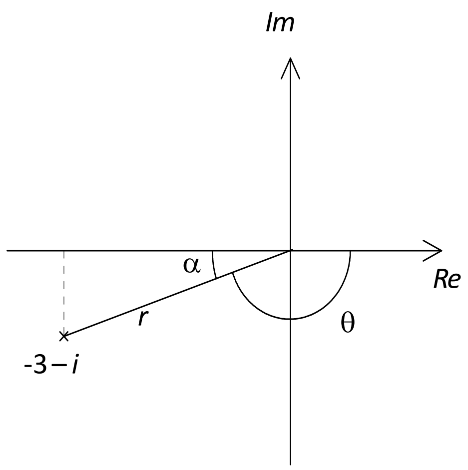


Note that in the diagram, θ_2 is negative. By convention, angles measured anticlockwise from the positive real axis are positive, and angles measured clockwise are negative.

Example

Calculate the modulus and argument of the complex number $-3-i$.

Solution



The modulus is $\sqrt{1^2+3^2}=\sqrt{10}=3.16$.

From basic trigonometry, we know that $\text{tangent}=\frac{\text{opposite}}{\text{adjacent}}$, so that $\tan\alpha=\frac{1}{3}$ *ie* $\alpha=\tan^{-1}\left(\frac{1}{3}\right)=0.322$ radians. So the angle, θ, will be given by $180^o=\pi$ radians less the 0.322 radians:

$$\pi-0.322=2.820 \text{ radians}$$

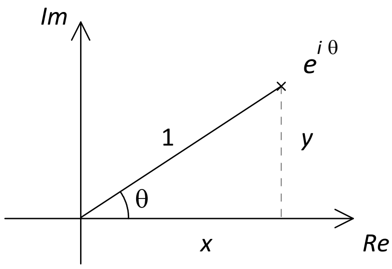
Recall that angles measured clockwise are negative, so the argument is $\theta=-2.820$ radians.

Question 1.3

Calculate the modulus and argument of the complex number $\dfrac{1-2i}{4-3i}$.

Euler's formula

As we have seen, complex numbers can be represented as points on an Argand diagram. The number $e^{i\theta}$ is represented by the point on the unit circle (the circle centred at the origin with radius 1) that forms an angle θ with the real axis.



We can work out the real and imaginary parts of this number (*ie* the horizontal and vertical co-ordinates) by drawing the triangle shown in the diagram. From basic trigonometry, we know that $\text{cosine} = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, so that $\cos\theta = \dfrac{x}{1}$ *ie* $x = \cos\theta$.

Similarly $y = \sin\theta$. So $e^{i\theta}$ has real part $\cos\theta$ and imaginary part $\sin\theta$. In other words:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

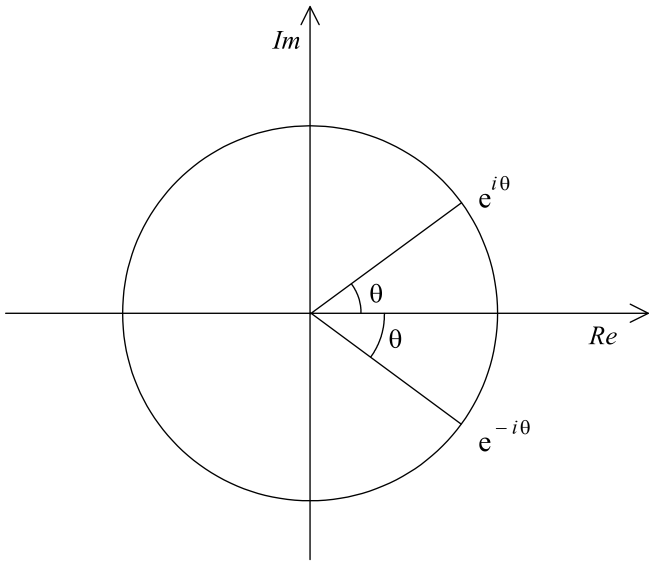
This is known as 'Euler's formula' and it holds for any value of θ (including complex ones). Note, however, that θ must be measured in radians not degrees.

> ***Question 1.4***
>
> What are the co-ordinates of the numbers $\exp\dfrac{i\pi}{4}$ and $2\exp\dfrac{i\pi}{6}$ on the Argand diagram?

Note that any complex number can be expressed in the form $re^{i\theta}$, which is known as polar form. θ is the argument and *r* is the modulus.

> ***Question 1.5***
>
> Express the complex number $7+5i$ in polar form.

It is useful to look at the complex conjugate of $e^{i\theta}$. The complex conjugate can be found by replacing *i* by $-i$ in the number, so here it is $e^{-i\theta}$. Since $e^{i\theta}=\cos\theta+i\sin\theta$, we can replace *i* by $-i$ to give $e^{-i\theta}=\cos\theta-i\sin\theta$.



Question 1.6

How can we deduce from this that $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$?

Question 1.7

Show that the product of $e^{-i\theta}$ and $e^{i\theta}$ does give a real number.

In many applied maths applications of complex numbers you will find that you need to add $e^{-i\theta}$ and $e^{i\theta}$. This gives:

$$e^{-i\theta}+e^{i\theta}=\cos\theta-i\sin\theta+\cos\theta+i\sin\theta=2\cos\theta$$

Note that this also gives a purely real number.

Question 1.8

Show that $(1+\frac{1}{2}e^{i\omega})(1+\frac{1}{2}e^{-i\omega})=\frac{5}{4}+\cos\omega$.

Solution of polynomial equations

If a polynomial has real coefficients, then any complex roots must occur in complex conjugate pairs. Allowing complex roots means that any quadratic equation will have possible solutions.

Example

Determine the roots of the equation $z^2 - 3z + 4 = 0$.

Solution

Using the quadratic formula, we get:

$$z = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{-7}}{2} = \frac{3 \pm \sqrt{(-1)}\sqrt{7}}{2} = \frac{3 \pm i\sqrt{7}}{2}$$

Simplifying this we get:

$$z = \tfrac{3}{2} + \tfrac{\sqrt{7}}{2}i \text{ or } \tfrac{3}{2} - \tfrac{\sqrt{7}}{2}i$$

Notice that these roots are a complex conjugate pair.

Question 1.9

You are given that $z = 1 + 2i$ is a root of the cubic equation $z^3 - 4z^2 + 9z - 10 = 0$. Find the other two roots.

Solutions

Solution 1.1

(i) $(2+3i)+(3-4i)=5-i$

(ii) $(4-6i)-(3i-7)=11-9i$

(iii) $(3-4i)(6+2i)=18-24i+6i-8i^2=26-18i$

(iv) $(4+2i)(4-2i)=16-8i+8i-4i^2=20$

Solution 1.2

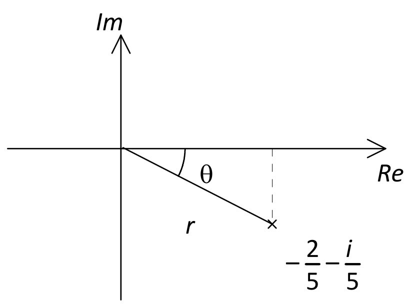
Multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{3i-4}{1-4i}\times\frac{1+4i}{1+4i}=\frac{3i-4-16i+12i^2}{1-16i^2}=\frac{-16-13i}{17}$$

Solution 1.3

We first need to simplify the complex number:

$$\frac{1-2i}{4-3i}\times\frac{4+3i}{4+3i}=\frac{4-8i+3i-6i^2}{16+9}=\frac{10-5i}{25}=\frac{2-i}{5}$$



The modulus is $\sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{1}{5}\right)^2}=0.4472$.

$\tan\theta = \dfrac{1/5}{2/5} = \frac{1}{2}$ *ie* $\theta = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$ radians.

Since angles measured clockwise are negative the argument is $\theta = -0.464$ radians.

Solution 1.4

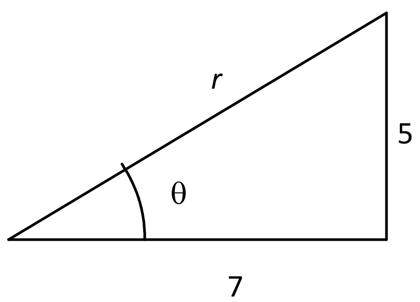
Using Euler's formula $\exp\dfrac{i\pi}{4} = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}$ so the co-ordinates are:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Similarly $2\exp\dfrac{i\pi}{6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2\left(\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right)$, so the co-ordinates are $(\sqrt{3}, 1)$.

Solution 1.5

Drawing a diagram, we have:



Using Pythagoras' Theorem the modulus is:

$$r = \sqrt{7^2 + 5^2} = \sqrt{74}$$

Using trigonometry, the argument is:

$$\tan\theta = \frac{5}{7} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{5}{7}\right) = 0.62025 \text{ radians}$$

So in polar form we have:

$$7+5i=\sqrt{74}e^{0.62025i}$$

Solution 1.6

We could work out $e^{-i\theta}$ by thinking of it as $e^{i(-\theta)}$ and applying the original form of Euler's formula:

$$e^{i(-\theta)}=\cos(-\theta)+i\sin(-\theta)$$

Comparing this with the other version for $e^{-i\theta}$, we see that $\cos(-\theta)$ must equal $\cos\theta$ and $\sin(-\theta)$ must equal $-\sin\theta$.

These properties of cosines and sines can be described by saying that cosine is an 'even' function and sine is an 'odd' function.

Solution 1.7

Consider their product:

$$\begin{aligned} e^{-i\theta}e^{i\theta} &= (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta) \\ &= \cos^2\theta - i\sin\theta\cos\theta + i\sin\theta\cos\theta - i^2\sin^2\theta \\ &= \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

which is a real number. You could, of course, have used the basic properties of powers to say that $e^{-i\theta}e^{i\theta}=e^{-i\theta+i\theta}=e^0=1$.

Solution 1.8

Multiplying out gives:

$$\begin{aligned}(1+\tfrac{1}{2}e^{i\omega})(1+\tfrac{1}{2}e^{-i\omega}) &= 1+\tfrac{1}{2}e^{i\omega}+\tfrac{1}{2}e^{-i\omega}+\tfrac{1}{2}e^{i\omega}\times\tfrac{1}{2}e^{-i\omega}\\ &= 1+\tfrac{1}{2}(e^{i\omega}+e^{-i\omega})+\tfrac{1}{4}\\ &= \tfrac{5}{4}+\tfrac{1}{2}\times 2\cos\omega\\ &= \tfrac{5}{4}+\cos\omega\end{aligned}$$

Solution 1.9

We are told that $1+2i$ is a root, and since the polynomial has real coefficients, we know that $1-2i$ must also be a root. The factors are therefore:

$$(z-(1+2i))(z-(1-2i))$$

and some other factor which is, as yet, unknown.

Multiplying these factors out we get:

$$(z-1-2i)(z-1+2i)=(z-1)^2-4i^2=z^2-2z+5$$

The cubic can then be written as:

$$z^3-4z^2+9z-10=(z^2-2z+5)(z-2)$$

Therefore the other two roots are 2 and $1-2i$.

