CORE READING

All of the Core Reading for the topics covered in this booklet is contained in this section.

Chapter 1 – Cashflow models

The practical work of the actuary often involves the management of various cashflows. These are simply sums of money, which are paid or received at different times. The timing of the cashflows may be known or uncertain. The amount of the individual cashflows may also be known or unknown in advance. From a theoretical viewpoint one may also consider a continuously payable cashflow.

For example, a company operating a privately owned bridge, road or tunnel will receive toll payments. The company will pay out money for maintenance, debt repayment and for other management expenses. From the company's viewpoint the toll payments are positive cashflows (i.e. money received) while the maintenance, debt repayments and other expenses are negative cashflows (*ie* money paid out). Similar cashflows arise in all businesses. In some businesses, such as insurance companies, investment income will be received in relation to positive cashflows (premiums) received before the negative cashflows (claims and expenses).

Where there is uncertainty about the amount or timing of cashflows, an actuary can assign probabilities to both the amount and the existence of a cashflow. In this Subject we will assume that the existence of the future cashflows is certain.

In this section, we provide examples of practical situations with cashflows that are assumed to be certain. In reality this may not be the case as the counterparty of a particular cashflow may not be able to pay out. For example, a company may fail and not be able to pay out interest on issued bonds.

1 **The term 'zero-coupon bond' is used to describe a security that is simply a contract to provide a specified lump sum at some specified future date. For the investor there is a negative cashflow at the point of investment and a single known positive cashflow on the specified future date.**

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2 **A body such as an industrial company, a local authority, or the government of a country may raise money by floating a loan on the stock exchange.**

In many instances such a loan takes the form of a fixed-interest security, which is issued in bonds of a stated nominal amount.

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3 **The characteristic feature of such a security in its simplest form is that the holder of a bond will receive a lump sum of specified amount at some specified future time together with a series of regular level interest payments until the repayment (or 'redemption') of the lump sum.**

The investor has an initial negative cashflow, a single known positive cashflow on the specified future date, and a series of smaller known positive cashflows on a regular set of specified future dates.

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4 **With a conventional fixed-interest security, the interest payments are all of the same amount. If inflationary pressures in the economy are not kept under control, the purchasing power of a given sum of money diminishes with the passage of time, significantly so when the rate of inflation is high. For this reason some investors are attracted by a security for which the actual cash amount of interest payments and of the final capital repayment are linked to an 'index' which reflects the effects of inflation.**

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5 **Here the initial negative cashflow is followed by a series of unknown positive cashflows and a single larger unknown positive cashflow, all on specified dates. However, it is known that the amounts of the future cashflows relate to the inflation index. Hence these cashflows are said to be known in 'real' terms.**

Note that in practice the operation of an index-linked security will be such that the cashflows do not relate to the inflation index at the time of payment, due to delays in calculating the index. It is also possible that the need of the borrower (or perhaps the investors) to know the amounts of the payments in advance may lead to the use of an index from an earlier period.

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6 If cash is placed on deposit, the investor can choose when to disinvest and will receive interest additions during the period of investment. The interest additions will be subject to regular change as determined by the investment provider. These additions may only be known on a day-to-day basis. The amounts and timing of cashflows will therefore be unknown.

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7 Equity shares (also known as 'shares' or 'equities' in the UK and as 'common stock' in the USA) are securities that are held by the owners of an organisation. Equity shareholders own the company that issued the shares. For example if a company issues 4,000 shares and an investor buys 1,000, the investor owns 25% of the company. In a small company all the equity shares may be held by a few individuals or institutions. In a large organisation there may be many thousands of shareholders.

Equity shares do not earn a fixed rate of interest as fixed-interest securities do. Instead the shareholders are entitled to a share in the company's profits, in proportion to the number of shares owned.

The distribution of profits to shareholders takes the form of regular payments of dividends. Since they are related to the company profits that are not known in advance, dividend rates are variable. It is expected that company profits will increase over time. It is therefore expected also that dividends per share will increase – though there are likely to be fluctuations. This means that in order to construct a cashflow schedule for an equity it is necessary first to make an assumption about the growth of future dividends. It also means that the entries in the cashflow schedule are uncertain – they are estimates rather than known quantities.

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8 In practice the relationship between dividends and profits is not a simple one. Companies will, from time to time, need to hold back some profits to provide funds for new projects or expansion. They may also hold back profits in good years to subsidise dividends in years with poorer profits. Additionally, companies may be able to distribute profits in a manner other than dividends, such as by buying back the shares issued to some investors.

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9 Since equities do not have a fixed redemption date, but can be held in perpetuity, we may assume that dividends continue indefinitely (unless the investor sells the shares or the company buys them back), but it is important to bear in mind the risk that the company will fail, in which case the dividend income will cease and the shareholders would only be entitled to any assets which remain after creditors are paid. The future positive cashflows for the investor are therefore uncertain in amount and may even be lower, in total, than the initial negative cashflow.

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10 An 'interest-only' loan is a loan that is repayable by a series of interest payments followed by a return of the initial loan amount.

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11 In the simplest of cases, the cashflows are the reverse of those for a fixed-interest security. The provider of the loan effectively buys a fixed-interest security from the borrower.

In practice, however, the interest rate need not be fixed in advance. The regular cashflows may therefore be of unknown amounts.

It may also be possible for the loan to be repaid early. The number of cashflows and the timing of the final cashflows may therefore be uncertain.

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12 A repayment loan is a loan that is repayable by a series of payments that include partial repayment of the loan capital in addition to the interest payments.

In its simplest form, the interest rate will be fixed and the payments will be of fixed equal amounts, paid at regular known times.

The cashflows are similar to those for an annuity certain.

As for the 'interest-only' loan, complications may be added by allowing the interest rate to vary or the loan to be repaid early. Additionally, it is possible that the regular repayments could be specified to increase (or decrease) with time. Such changes could be smooth or discrete.

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13 It is important to appreciate that with a repayment loan the breakdown of each payment into 'interest' and 'capital' changes significantly over the period of the loan. The first repayment will consist almost entirely of interest and will provide only a very small capital repayment. In contrast, the final repayment will consist almost entirely of capital and will have a small interest content.

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14 An annuity certain provides a series of regular payments in return for a single premium (*ie* a lump sum) paid at the outset. The precise conditions under which the annuity payments will be made will be clearly specified. In particular, the number of years for which the annuity is payable, and the frequency of payment, will be specified. Also, the payment amounts may be level or might be specified to vary – for example in line with an inflation index, or at a constant rate.

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15 The cashflows for the investor will be an initial negative cashflow followed by a series of smaller regular positive cashflows throughout the specified term of payment. In the case of level annuity payments, the cashflows are similar to those for a fixed-interest security.

From the perspective of the annuity provider, there is an initial positive cashflow followed by a known number of regular negative cashflows.

The theory can be extended to deal with annuities where the payment term is uncertain, that is, for which payments are made only so long as the annuity policyholder survives.

Insurance contracts

16 **The cashflows for the examples covered in this section differ than the previous in that the frequency, severity, and/or timing of the cashflow may be unkown. For example, a typical cover of a life cover may have a specified date on which a pre-agreed amount is paid on survival – but the benefit payment may not be paid if the individual does not survive.**

Similarly, a pension pays out a known amount at a specified time per month, but only if the individual is alive. Typically the severity is known and pre-specified in life-insurance contracts.

On the other hand, a non-life (general) insurance cover tends to not have known severities. For example, the cost of a car accident may range from a few pounds in the case of a small collision to millions in case of a major accident that caused death.

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17 **A pure endowment is an insurance policy which provides a lump sum benefit on survival to the end of a specified term usually in return for a series of regular premiums. In some cases a lump-sum premium is paid. In this case, the cashflows for the policyholder will be a negative cashflow at inception and a positive cashflow at the end of term, only if the policyholder has survived.**

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier. A large, positive cashflow occurs at the end of the term, only if the policyholder has survived. If the policyholder dies before the end of the term there is no positive cashflow.

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow, contingent on policyholder survival.

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18 **An endowment assurance is similar in that it provides a survival benefit at the end of the term, but it also provides a lump sum benefit on death before the end of the term. The benefits are provided in return for a series of regular premiums**

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death, if earlier, followed by a large positive cashflow at the end of the term (or death, if earlier). Depending on the terms of the policy, the amount payable on death may not be the same as that payable on survival.

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow. The negative cashflow is certain to be paid, but the timing of that payment depends on whether/when the policyholder dies.

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19 A term assurance is an insurance policy which provides a lump sum benefit on death before the end of a specified term usually in return for a series of regular premiums.

The cashflows for the policyholder will be a series of negative cashflows throughout the specified term or until death (or one negative cashflow at inception if paid on a lump-sum basis), if earlier, followed by a large positive cashflow payable on death, if death occurs before the end of the term. If the policyholder survives to the end of the term there is no positive cashflow.

From the perspective of the insurer, there is a stream of regular positive cashflows which cease at a specified point (or earlier, if the policyholder dies) followed by a large negative cashflow, contingent on policyholder death during the term.

Generally, the negative cashflow (death benefit), if it occurs, is significantly higher than the positive cashflow (premiums), when compared to, say, a pure endowment. This is because, for each individual policy, the probability of the benefit being paid is generally lower than for endowments because it is contingent on death, rather than on survival.

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20 A contingent annuity is a similar contract to the annuity certain but the payments are contingent upon certain events, such as survival, hence the payment term for the regular cashflows (which will be negative from the perspective of the annuity provider) is uncertain.

Typical examples of contingent annuities include:

- **a single life annuity – where the regular payments made to the annuitant are contingent on the survival of that annuitant.**
- **a joint life annuity – which covers two lives, where the regular payments are contingent on the survival of one or both of those lives.**
- **a reversionary annuity – which is based on two lives, where the regular payments start on the death of the first life if, and only if, the second life is alive at the time. Payments then continue until the death of the second life.**

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21 A typical car insurance contract lasts for one year. In return for a premium which can be paid as a single lump sum or at monthly intervals, the insurer will provide cover to pay for damage to the insured vehicle or fire or theft of the vehicle, known as 'property cover'. In many countries, such as the UK, the contract also provides cover for compensation payable to third parties for death, injury or damage to their property, known as 'liability cover'.

Depending on the terms of the policy, the insurance company may settle claims directly with the policyholder or with another party. For example, in the case of theft or total loss, the insurance company may pay a lump sum to the policyholder in lieu of that loss. In the case of damage to the insured vehicle the insurance company may settle the claim directly with the party undertaking the repairs without involving the policyholder. In the case of third party liability claims the insurance company may settle the claims directly with the third party.

In some cases, the policyholder may be required to cover the cost of damage or repairs first before the insurance company settles the claim, in which case the insurance company will pay the policyholder directly.

The cashflows for the policyholder will usually be a single negative cashflow at the beginning of the year. Further cashflows only take place in the event of a claim. If the policyholder has to pay for repairs or compensation, this will incur a further negative cashflow, followed by a positive cashflow when the insurance company settles the claim. If the insurance company settles the claim directly with the repair company or third party, the policyholder may not experience further cashflows.

From the insurer's perspective there will be a positive cashflow at the beginning of the policy, followed by a negative cashflow when the claim is settled.

The timing of the cashflows will depend on how long the claim takes to be reported and settled. Typically property claims take less time to settle than liability claims. Where liability claims involve disputes, for example necessitating court judgements, they can take years to settle and the amounts are less certain.

Cashflows tend to be short term and are payable within the year.

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22 A typical health insurance contract lasts for one year. In return for a premium, the policyholder is entitled to benefits which may include hospital treatment either paid for in full or in part, and/or cash benefits in lieu of treatment, such as a fixed sum per day spent in hospital as an in-patient.

From the policholder's perspective the cashflows will include a negative cashflow at the beginning of the year followed by positive cashflows in the event of a claim in the case of a cash benefit. Where the insurance company pays for hospital treatment directly, the policyholder may experience no more cashflows after paying the initial premium.

From the perspective of the insurer, there will be an initial positive cashflow at the start of the policy followed by negative cashflows in the event of a claim, when those claims are settled.

Cashflows tend to be short term and are payable within the year.

Chapter 4 – The time value of money

23 ***Interest*** **may be regarded as a reward paid by one person or organisation (the *borrower*) for the use of an asset, referred to as capital, belonging to another person or organisation (the *lender*).**

When the capital and interest are expressed in monetary terms, capital is also referred to as *principal*. The total received by the lender after a period of time is called the *accumulated value*. The difference between the principal and the accumulated value is called the *interest*. Note that we are assuming here that no other payments are made or incurred (*eg* charges, expenses).

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24 **If there is some risk of default (*ie* loss of capital or non-payment of interest) a lender would expect to be paid a higher rate of interest than would otherwise be the case.**

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25 **Another factor that may influence the rate of interest on any transaction is an allowance for the possible depreciation or appreciation in the value of the currency in which the transaction is carried out. This factor is very important in times of high inflation.**

Interest

26 **The essential feature of simple interest is that interest, once credited to an account, does not itself earn further interest.**

27 **Suppose an amount C is deposited in an account that pays simple interest at the rate of $i \times 100\%$ per annum. Then after n years the deposit will have accumulated to:**

$$C(1+ni) \qquad \textbf{(1.1)}$$

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28 **When n is not an integer, interest is paid on a pro-rata basis.**

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29 **The essential feature of compound interest is that interest itself earns interest.**

30 **Suppose an amount C is deposited in an account that pays compound interest at the rate of $i \times 100\%$ per annum. Then after n years the deposit will have accumulated to:**

$$C(1+i)^n \tag{1.2}$$

31 **For $t_1 \le t_2$ we define $A(t_1, t_2)$ to be the accumulation at time t_2 of an investment of 1 at time t_1.**

32 **The number $A(t_1, t_2)$ is often called an *accumulation factor*, since the accumulation at time t_2 of an investment of C at time t_1 is, by proportion:**

$$CA(t_1, t_2) \tag{1.3}$$

33 **$A(n)$ is often used as an abbreviation for the accumulation factor $A(0, n)$.**

34 **Now let $t_0 \le t_1 \le t_2$ and consider an investment of 1 at time t_0. The proceeds at time t_2 will be $A(t_0, t_2)$ if one invests at time t_0 for term $t_2 - t_0$, or $A(t_0, t_1)A(t_1, t_2)$ if one invests at time t_0 for term $t_1 - t_0$ and then, at time t_1, reinvests the proceeds for term $t_2 - t_1$. In a consistent market these proceeds should not depend on the course of action taken by the investor. Accordingly, we say that under the principle of consistency:**

$$A(t_0, t_n) = A(t_0, t_1)A(t_1, t_2) \ldots A(t_{n-1}, t_n) \tag{1.4}$$

Present values

35 It follows by formula (1.2) that an investment of:

$$C/(1+i)^n \tag{2.1}$$

at time 0 (the present time) will give C at time $n \geq 0$.

This is called the *discounted present value* (or, more briefly, the *present value*) of C due at time $n \geq 0$.

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36 We now define the function:

$$v = 1/(1+i) \tag{2.2}$$

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37 It follows It follows by formulae (2.1) and (2.2) that the discounted present value of C due at time $n \geq 0$ is:

$$Cv^n \tag{2.3}$$

Discount rates

38 An alternative way of obtaining the discounted value of a payment is to use discount rates.

As has been seen with simple interest, the interest earned is not itself subject to further interest. The same is true of simple discount, which is defined below.

Suppose an amount C is due after n years and a rate of *simple discount* of d per annum applies. Then the sum of money required to be invested now to amount to C after n years (*ie* the present value of C) is:

$$C(1-nd) \tag{3.1}$$

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39 **In normal commercial practice, d is usually encountered only for periods of less than a year. If a lender bases his short-term transactions on a simple rate of discount d then, in return for a repayment of X after a period t $(t < 1)$ he will lend $X(1 - td)$ at the start of the period. In this situation, d is also known as a rate of *commercial discount*.**

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40 **As has been seen with compound interest, the interest earned is subject to further interest. The same is true of compound discount, which is defined below.**

Suppose an amount C is due after n years and a rate of compound (or effective) discount of d per annum applies. Then the sum of money required to be invested now to accumulate to C after n years (*ie* the present value of C) is:

$$C(1-d)^n \qquad \textbf{(3.2)}$$

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41 **In the same way that the accumulation factor $A(n)$ gives the accumulation at time n of an investment of 1 at time 0, we define $v(n)$ to be the present value of a payment of 1 due at time n. Hence:**

$$v(n) = \frac{1}{A(n)} \qquad \textbf{(3.3)}$$

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42 **Effective rates are compound rates that have interest paid *once* per unit time either at the end of the period (effective interest) or at the beginning of the period (effective discount). This distinguishes them from nominal rates where interest is paid more frequently than once per unit time.**

We can demonstrate the equivalence of compound and effective rates by an alternative way of thinking about effective rates.

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43 An investor will lend an amount 1 at time 0 in return for a repayment of $(1+i)$ at time 1. Hence we can consider i to be the interest paid at the *end* of the year. Accordingly i is called the *rate of interest* (or the *effective rate of interest*) per unit time.

So denoting the effective rate of interest during the n th period by i_n, we have:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} \qquad (4.1)$$

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44 If i is the compound rate of interest, we have:

$$i_n = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = (1+i) - 1 = i \qquad (4.2)$$

Since this is independent of n, we see that the effective rate of interest is identical to the compound rate of interest we met earlier

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45 We can think of compound discount as an investor lending an amount $(1-d)$ at time 0 in return for a repayment of 1 at time 1. The sum of $(1-d)$ may be considered as a loan of 1 (to be repaid after 1 unit of time) on which interest of amount d is payable in advance. Accordingly d is called the *rate of discount* (or the *effective rate of discount*) per unit time.

We can also show that the effective rate of discount is identical to the compound rate of discount we met earlier.

Equivalent rates

46 Two rates of interest and/or discount are equivalent if a given amount of principal invested for the same length of time produces the same accumulated value under each of the rates.

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47 **Comparing formulae (2.3) and (3.2), we see that:**

$$v = 1 - d \tag{5.1}$$

And from (2.2) and (5.1) we obtain the rearrangements:

$$d = iv \tag{5.2}$$

and: $$d = \frac{i}{1+i} \tag{5.3}$$

Recall that d is the interest paid at time 0 on a loan of 1, whereas i is the interest paid at time 1 on the same loan. If the rates are equivalent then if we discount i from time 1 to time 0 we will obtain d. This is the interpretation of equations (5.2) and (5.3).

Chapter 5 – Interest rates

Nominal rates of interest and discount

48 **Recall from earlier that 'effective' rates of interest and discount have interest paid once per measurement period, either at the end of the period or at the beginning of the period.**

'Nominal' is used where interest is paid more (or less) frequently than once per measurement period.

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49 **We denote the nominal rate of interest payable *p* times per period by $i^{(p)}$. This is also referred to as the rate of interest convertible *p*thly or compounded *p*thly.**

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50 **A nominal rate of interest per period, payable pthly, $i^{(p)}$, is defined to be a rate of interest of $i^{(p)}/p$ applied for each pth of a period. For example, a nominal rate of interest of 6% *pa* convertible quarterly means an interest rate of $6/4 = 1.5\%$ per quarter.**

Hence, by definition, $i^{(p)}$ is equivalent to a *p*thly effective rate of interest of $i^{(p)}/p$.

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51 Therefore the effective interest rate i is obtained from:

$$1+i=\left(1+\frac{i^{(p)}}{p}\right)^{p} \tag{6.1}$$

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52 Note that $i^{(1)}=i$.

The treatment of problems involving nominal rates of interest (or discount) is almost always considerably simplified by an appropriate choice of the time unit.

By choosing the basic time unit to be the period corresponding to the frequency with which the nominal rate of interest is convertible, we can use $i^{(p)}/p$ as the effective rate of interest per unit time. For example, if we have a nominal rate of interest of 18% per annum convertible monthly, we should take one month as the unit of time and 1½% as the rate of interest per unit time.

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53 We denote the nominal rate of discount payable p times per period by $d^{(p)}$. This is also referred to as the rate of discount convertible pthly or compounded pthly.

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54 A nominal rate of discount per period payable pthly, $d^{(p)}$, is defined as a rate of discount of $d^{(p)}/p$ applied for each *p*th of a period.

Hence, by definition, $d^{(p)}$ is equivalent to a *p*thly effective rate of discount of $d^{(p)}/p$.

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55 Therefore the effective discount rate d is obtained from:

$$1-d=\left(1-\frac{d^{(p)}}{p}\right)^{p} \qquad \textbf{(6.2)}$$

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56 Note that $d^{(1)}=d$.

The force of interest

57 We assume that for each value of i there is number, δ , such that:

$$\lim_{p\to\infty} i^{(p)}=\delta$$

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58 δ is the nominal rate of interest per unit time convertible continuously (or momently). This is also referred to as the rate continuously compounded. We call it the *force of interest*.

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59 Euler's rule states that:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}=e^{x}$$

Applying this to the right-hand-side of (6.1) gives:

$$\lim_{p\to\infty}\left(1+\frac{i^{(p)}}{p}\right)^{p}=e^{i^{(\infty)}}$$

Hence:

$$1+i=e^{\delta} \qquad \textbf{(7.1)}$$

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60 **Since $v = (1+i)^{-1}$, we have:**

$$v = e^{-\delta} \qquad \textbf{(7.2)}$$

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61 **From equation (7.2) we have:**

$$v^t = (e^{-\delta})^t = e^{-\delta t}$$

Hence, the discount factor for a force of interest δ is:

$$v(n) = e^{-\delta n}$$

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62 **It can also be shown that:**

$$\lim_{p\to\infty} d^{(p)} = \delta$$

However, $d^{(p)}$ tends to this limit from below whereas $i^{(p)}$ tends to this limit from above.

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63 **Hence, we have:**

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i$$

Relationships between effective, nominal and force of interest

64 **Recall that effective interest i can be thought of as interest paid at the end of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of $(1+i)$ at time 1.**

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65 **Similarly, nominal interest convertible *p*thly can be thought of as the total interest per unit of time paid on a loan of amount 1 at time 0, where interest is paid in p equal instalments at the end of each *p*th subinterval (*ie* at times $1/p, 2/p, 3/p, \ldots, 1$).**

Since $i^{(p)}$ is the *total* interest paid and each interest payment is of amount $i^{(p)}/p$ then the accumulated value at time 1 of the interest payments is:

$$\frac{i^{(p)}}{p}(1+i)^{(p-1)/p}+\frac{i^{(p)}}{p}(1+i)^{(p-2)/p}+\cdots+\frac{i^{(p)}}{p}=i$$

Hence:

$$i^{(p)}=p\left[(1+i)^{1/p}-1\right]$$

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66 Recall that effective discount d can be thought of as interest paid at the start of the period. Hence, an investor lending an amount 1 at time 0 receives a repayment of 1 at time 1, but d is paid at the start so a sum of $(1-d)$ is lent at time 0.

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67 Similarly, $d^{(p)}$ is the total amount of interest per unit of time payable in equal instalments at the start of each *p*th subinterval (*ie* at times $0, 1/p, 2/p, \ldots, (p-1)/p$).

As a consequence the present value at time 0 of the interest payments is:

$$\frac{d^{(p)}}{p}+\frac{d^{(p)}}{p}(1-d)^{1/p}+\cdots+\frac{d^{(p)}}{p}(1-d)^{(p-1)/p}=d$$

Hence:

$$d^{(p)}=p\left[1-(1-d)^{1/p}\right]$$

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68 Now δ is the total amount of interest payable as a continuous payment stream, *ie* an amount δdt is paid over an infinitesimally small period dt at time t .

As a consequence the accumulated value at time 1 of these interest payments is:

$$\int_0^1 \delta (1+i)^{1-t}\, dt$$

which, by symmetry, is equal to:

$$\int_0^1 \delta (1+i)^{t}\, dt = i$$

Hence:

$$\delta = \ln(1+i) \qquad \text{or} \qquad e^{\delta} = 1+i$$

It is essential to appreciate that, at force of interest δ per unit time, the five series of payments illustrated in Figure 1 below all have the same value.

	0	$\frac{1}{p}$	$\frac{2}{p}$	$\frac{3}{p}$	...	$\frac{p-1}{p}$	1	time
(1)	d							
(2)	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	$\frac{d^{(p)}}{p}$	...	$\frac{d^{(p)}}{p}$		
(3)		$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	...	$\frac{i^{(p)}}{p}$	$\frac{i^{(p)}}{p}$	equivalent payments
(4)							i	
(5)	←			δ			→	

Figure 1 Equivalent payments

Force of interest as a function of time

69 The force of interest is the instantaneous change in the fund value, expressed as an annualized percentage of the current fund value.

So the force of interest at time t is defined to be:

$$\delta(t) = \frac{V_t'}{V_t}$$

where V_t is the value of the fund at time t and V_t' is the derivative of V_t with respect to t.

Hence:

$$\delta(t) = \frac{d}{dt}\ln V_t$$

Integrating this from t_1 to t_2 gives:

$$\int_{t_1}^{t_2} \delta(t)dt = \left[\ln V_t\right]_{t_1}^{t_2} = \ln V_{t_2} - \ln V_{t_1} = \ln\left(\frac{V_{t_2}}{V_{t_1}}\right)$$

$$\Rightarrow \quad \frac{V_{t_2}}{V_{t_1}} = \exp\left(\int_{t_1}^{t_2} \delta(t)\, dt\right)$$

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70 Hence:

$$A(t_1,t_2) = \exp\left(\int_{t_1}^{t_2} \delta(t)\, dt\right)$$

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71 For the case when the force of interest is constant, δ , between time 0 and time n , we have:

$$A(0,n) = e^{\int_0^n \delta \, dt} = e^{\delta n}$$

Hence:

$$(1+i)^n = e^{\delta n}$$

Therefore:

$$(1+i) = e^{\delta}$$

as before.

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72 Although the force of interest is a theoretical measure it is the most fundamental measure of interest (as all other interest rates can be derived from it). However, since the majority of transactions involve discrete processes we tend to use other interest rates in practice.

It still remains a useful conceptual and analytical tool and can be used as an approximation to interest paid very frequently, *eg* daily.

Chapter 7 – Discounting and accumulating

Present values of cashflows

73 In many compound interest problems one must find the discounted present value of cashflows due in the future. It is important to distinguish between (a) discrete and (b) continuous payments.

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74 **The present value of the sums $c_{t_1}, c_{t_2}, ..., c_{t_n}$ due at times $t_1, t_2, ..., t_n$ (where $0 \le t_1 < t_2 < \ldots < t_n$) is:**

$$c_{t_1} v(t_1) + c_{t_2} v(t_2) + \cdots + c_{t_n} v(t_n) = \sum_{j=1}^{n} c_{t_j} v(t_j)$$

If the number of payments is infinite, the present value is defined to be:

$$\sum_{j=1}^{\infty} c_{t_j} v(t_j)$$

provided that this series converges. It usually will in practical problems.

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75 **Suppose that $T > 0$ and that between times 0 and T an investor will be paid money continuously, the rate of payment at time t being £ $\rho(t)$ per unit time. What is the present value of this cashflow?**

In order to answer this question it is essential to understand what is meant by the rate of payment of the cashflow at time t. If M(t) denotes the total payment made between time 0 and time t, then by definition:

$$\rho(t) = M'(t) \text{ for all t}$$

Then, if $0 \le \alpha \le \beta \le T$, the total payment received between time α and time β is:

$$M(\beta) - M(\alpha) = \int_{\alpha}^{\beta} M'(t)dt = \int_{\alpha}^{\beta} \rho(t)dt \qquad (8.1)$$

Thus the rate of payment at any time is simply the derivative of the total amount paid up to that time, and the total amount paid between any two times is the integral of the rate of payments over the appropriate time interval.

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76 Between times t and $t+dt$ the total payment received is $M(t+dt)-M(t)$. If dt is very small this is approximately $M'(t)dt$ or $\rho(t)dt$. Theoretically, therefore, we may consider the present value of the money received between times t and $t+dt$ as $v(t)\rho(t)dt$. The present value of the entire cashflow is obtained by integration as:

$$\int_0^T v(t)\rho(t)dt$$

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77 If T is infinite we obtain, by a similar argument, the present value:

$$\int_0^\infty v(t)\rho(t)dt$$

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78 By combining the results for discrete and continuous cashflows, we obtain the formula:

$$\sum c_t v(t) + \int_0^\infty v(t)\rho(t)dt \qquad (8.2)$$

for the present value of a general cashflow (the summation being over those values of t for which c_t, the discrete cashflow at time t, is non-zero).

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79 So far we have assumed that all payments, whether discrete or continuous, are positive. If one has a series of income payments (which may be regarded as positive) and a series of outgoings (which may be regarded as negative) their *net present value* is defined as the difference between the value of the positive cashflow and the value of the negative cashflow.

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80 **Consider times t_1 and t_2, where t_2 is not necessarily greater than t_1. The value at time t_1 of the sum C due at time t_2 is defined as:**

(a) If $t_1 \geq t_2$, the accumulation of C from time t_2 until time t_1; or

(b) If $t_1 < t_2$, the discounted value at time t_1 of C due at time t_2.

In both cases the value at time t_1 of C due at time t_2 is:

$$C\exp\left[-\int_{t_1}^{t_2}\delta(t)dt\right] \tag{9.1}$$

(Note the convention that, if $t_1 > t_2$, $\int_{t_1}^{t_2}\delta(t)dt = -\int_{t_2}^{t_1}\delta(t)dt$.)

Since:

$$\int_{t_1}^{t_2}\delta(t)dt = \int_0^{t_2}\delta(t)dt - \int_0^{t_1}\delta(t)dt$$

it follows immediately from Equation (9.1) that the value at time t_1 of C due at time t_2 is:

$$C\,\frac{v(t_2)}{v(t_1)} \tag{9.2}$$

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81 **The value at a general time t_1 of a discrete cashflow of c_t at time t (for various values of t) and a continuous payment stream at rate $\rho(t)$ per time unit may now be found, by the methods given earlier, as:**

$$\sum c_t\,\frac{v(t)}{v(t_1)} + \int_{-\infty}^{\infty}\rho(t)\frac{v(t)}{v(t_1)}dt \tag{9.3}$$

where the summation is over those values of t for which $c_t \neq 0$.

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82 **We note that in the special case when $t_1 = 0$ (the present time), the value of the cashflow is:**

$$\sum c_t v(t) + \int_{-\infty}^{\infty} \rho(t)\, v(t)\, dt$$

where the summation is over those values of t for which $c_t \neq 0$.

This is a generalisation of formula (8.2) to cover the past as well as present or future payments. If there are incoming and outgoing payments, the corresponding *net value* may be defined, as earlier, as the difference between the value of the positive and the negative cashflows. If all the payments are due at or after time t_1, their value at time t_1 may also be called their 'discounted value', and if they are due at or before time t_1, their value may be referred to as their 'accumulation'.

It follows that any value may be expressed as the sum of a discounted value and an accumulation. This fact is helpful in certain problems. Also, if $t_1 = 0$ and all the payments are due at or after the present time, their value may also be described as their '(discounted) present value', as defined by formula (8.2).

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83 **It follows from formula (9.2) that the value at any time t_1 of a cashflow may be obtained from its value at another time t_2 by applying the factor $v(t_2)/v(t_1)$, *ie*:**

$$\begin{bmatrix}\text{Value at time } t_1 \\ \text{of cash flow}\end{bmatrix} = \begin{bmatrix}\text{Value at time } t_2 \\ \text{of cash flow}\end{bmatrix} \begin{bmatrix}\dfrac{v(t_2)}{v(t_1)}\end{bmatrix}$$

or:

$$\begin{bmatrix}\text{Value at time } t_1 \\ \text{of cash flow}\end{bmatrix} \big[v(t_1)\big] = \begin{bmatrix}\text{Value at time } t_2 \\ \text{of cash flow}\end{bmatrix} \big[v(t_2)\big] \qquad \textbf{(9.4)}$$

Each side of Equation (9.4) is the value of the cashflow at the present time (time 0).

In particular, by choosing time t_2 as the present time and letting $t_1 = t$, we obtain the result:

$$\begin{bmatrix}\text{Value at time } t \\ \text{of cash flow}\end{bmatrix} = \begin{bmatrix}\text{Value at the present} \\ \text{time of cash flow}\end{bmatrix}\left[\frac{1}{v(t)}\right]$$

These results are useful in many practical examples. The time 0 and the unit of time may be chosen so as to simplify the calculations.

Interest income

84 Consider now an investor who wishes not to accumulate money but to receive an income while keeping his capital fixed at *C*. If the rate of interest is fixed at *i* per time unit, and if the investor wishes to receive income at the end of each time unit, it is clear that the income will be *iC* per time unit, payable in arrear, until such time as the capital is withdrawn.

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85 However, if interest is paid continuously with force of interest $\delta(t)$ at time t then the income received between times t and $t+dt$ will be $C\delta(t)dt$.

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86 So the total interest income from time 0 to time T will be:

$$I(T) = \int_0^T C\delta(t)\,dt$$

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87 If the investor withdraws the capital at time T, the present values of the income and capital at time 0 are:

$$C\int_0^T \delta(t)v(t)dt \quad \text{and} \quad Cv(T)$$

respectively.