Subject CM2

CMP Upgrade 2023/24

CMP Upgrade

This CMP Upgrade lists the changes to the Syllabus, Core Reading and the ActEd material since last year that might realistically affect your chance of success in the exam. It is produced so that you can manually amend your 2023 CMP to make it suitable for study for the 2024 exams. It includes replacement pages and additional pages where appropriate.

Alternatively, you can buy a full set of up-to-date Course Notes / CMP at a significantly reduced price if you have previously bought the full-price Course Notes / CMP in this subject. Please see our 2024 *Student Brochure* for more details.

We only accept the current version of assignments for marking, *ie* those published for the sessions leading to the 2024 exams. If you wish to submit your script for marking but only have an old version, then you can order the current assignments free of charge if you have purchased the same assignments in the same subject in a previous year, and have purchased marking for the 2024 session.

This CMP Upgrade contains:

- all significant changes to the Syllabus and Core Reading
- additional changes to the ActEd Course Notes and Assignments that will make them suitable for study for the 2024 exams.

1 Changes to the Syllabus

The wording of all the syllabus items has been changed slightly and some syllabus items have been combined.

The syllabus items corresponding to Chapter 3 (Stochastic dominance and behavioural finance) and Chapter 5 (Stochastic models of investment returns) have been removed from the course. As such these two chapters have been removed from the course and all the other chapters have been renumbered accordingly.

The CM2 syllabus now reads as follows:

Syllabus

The Syllabus for Subject CM2 is given here. To the right of each objective are the chapter numbers in which the objective is covered in the ActEd course.

Aim

Economic Modelling (Subject CM2) provides a grounding in the principles of actuarial modelling, focusing on stochastic asset models, the valuation of financial derivatives and develops skills to model economic decision making, the probability of ruin, estimation of claims and the pricing of assets and options.

Topics and topic weightings

This subject covers the following topics:

1. Rational economic theory (10%)
2. Measures of investment risk (10%)
3. Asset valuations (30%)
4. Liability valuations (20%)
5. Option theory (30%)

The topic weighting percentage noted alongside the topics is indicative of the volume of content of a topic within the subject and therefore broadly aligned to the volume of marks allocated to this topic in the examination. For example, if a topic is 20% of the subject then you can expect that approximately 20% of the total marks available in the examination paper will be available on that topic.

Students should ensure that they are well prepared across the entire syllabus and have an understanding of the principal terms used within the course.

Objectives

1 Rational economic theory (10%)

Theories and modelling techniques used to explore, understand and evaluate rational economic decision making and asset pricing. In particular, the application of utility functions to financial and economic problems.

1.1 Understand the principles of rational expectations theory (Chapter 1)

1.1.1 Three forms of the Efficient Markets Hypothesis and their consequences for investment management

1.1.2 Evidence for or against each form of the Efficient Markets Hypothesis

1.2 Understand the principles of rational choice theory (Chapter 2)

1.2.1 Meaning of 'utility function'

1.2.2 Concept of utility theory and the expected utility theorem

1.2.3 Understand properties of utility functions that express these economic characteristics of investors:

- Non-satiation
- Risk aversion, risk neutrality and risk seeking
- Declining or increasing absolute and relative risk aversion

1.2.4 Economic properties of commonly used utility functions

1.2.5 Identify how a utility function may depend on current wealth and discuss state-dependent utility functions

1.2.6 Perform calculations using common utility functions that compare investment opportunities

1.2.7 Use utility theory to analyse simple insurance problems

2 Measures of investment risk (10%) (Chapter 3)

Apply a range of financial risk measurement tools to evaluate investment opportunities in the context of utility functions. Understand how mitigating actions can reduce risk faced by insurance companies.

2.1 Identify the properties of risk measures and use these risk measures to compare and analyse investment opportunities

2.1.1 Measures of investment risk:

- Variance of return
- Downside semi-variance of return
- Shortfall probabilities
- Value at Risk (VaR)
- TailVaR (also referred to as Expected Shortfall)

2.1.2 How the risk measures listed in are related to the form of an investor's utility function

2.1.3 Compare investment opportunities via calculations using the risk measures listed in 2.1.1

2.1.4 How the distribution of returns and the thickness of tails will influence the assessment of risk

2.2 The role of insurance companies in reducing or removing risk

2.2.1 How insurance companies help to reduce or remove risk

2.2.2 The meaning of 'moral hazard' and 'adverse selection'

3 Asset valuations (30%)

The use of models in portfolio selection and asset pricing, including the term structure of interest rates and credit risk.

3.1 Understand mean-variance portfolio theory and its application

3.1.1 The assumptions of mean-variance portfolio theory (Chapter 4)

3.1.2 When does the application of mean-variance portfolio theory lead to the selection of an optimum portfolio

3.1.3 Use mean-variance portfolio theory to calculate the expected return and risk of a portfolio of many risky assets, given the expected return, variance and covariance of returns of the individual assets

3.1.4 Benefits of diversification using mean-variance portfolio theory

3.2 Understand and use the Capital Asset Pricing Model (CAPM) (Chapter 6)

3.2.1 The assumptions, principal results and uses of the Sharpe-Lintner-Mossin Capital Asset Pricing Model (CAPM)

3.2.2 he limitations of the basic CAPM and some of the attempts that have been made to develop the theory to overcome these limitation

3.2.3 Perform calculations using the CAPM

3.2.4 Main issues of estimating parameters for asset pricing models

3.3 Understand and use single and multifactor models for investment returns (Chapter 5)

3.3.1 Three types of multifactor models of asset returns

- Macroeconomic models
- Fundamental factor models
- Statistical factor models

3.3.2 Single-index model of asset returns

3.3.3 Concepts of diversifiable and non-diversifiable risk

3.3.4 Construction of the different types of multifactor models

3.3.5 Perform calculations using both single and multifactor models for investment returns

3.4 Appreciate different stochastic models for security prices and how and when they can be applied (Chapters 7, 8 and 9)

3.4.1 Continuous time log-normal model of security prices and the empirical evidence for and against the model

3.4.2 Basic properties of standard Brownian motion or Wiener process

3.4.3 Principles of stochastic differential equations, the Ito integral, diffusion and mean-reverting processes

3.4.4 Understand Ito's Lemma and apply it to simple problems

3.4.5 Describe the stochastic differential equation for geometric Brownian motion

3.4.6 Describe the stochastic differential equation for the Ornstein-Uhlenbeck process

3.5 Understand the principles and characteristics of models of the term structures of interest rates and their application (Chapter 16)

3.5.1 Principal concepts and terms underlying the theory of a term structure of interest rates

3.5.2 Desirable characteristics of models for the term structure of interest rates

3.5.3 Apply the term structure of interest rates to modelling various cashflows

3.5.4 Risk-neutral approach to the pricing of zero-coupon bonds and interest-rate derivatives for a general one-factor diffusion model for the risk-free rate of interest, as a computational tool

3.5.5 The Vasicek, Cox-Ingersoll-Ross and Hull-White models for the term structure of interest rates and their limitations

3.6 Understand the principles and application of simple models for credit risk (Chapter 17)

3.6.1 What is a 'credit event' and 'recovery rate'

3.6.2 Identify the different approaches to modelling credit risk: structural models, reduced form models, intensity-based models

3.6.3 Understand and apply the Merton model

3.6.4 Understand and apply the two-state model for credit rating with a constant transition intensity

3.6.5 Generalisation of the two-state model:

- To the Jarrow-Lando-Turnbull model for credit ratings
- To incorporate a stochastic transition intensity

4 Liability Valuations (20%)

The use of models in insurance to calculate the probability of ruin and estimate claims.

4.1 Understand the principles and application of ruin theory (Chapter 18)

4.1.1 The aggregate claim process and the cashflow process for a risk

4.1.2 Use the Poisson process and the distribution of inter-event times to calculate probabilities of the number of events in a given time interval and waiting times

4.1.3 Understand the compound Poisson process and calculate probabilities using simulation

4.1.4 The probability of ruin in infinite/finite and continuous/discrete time and state, and the relationships between the different probabilities of ruin

4.1.5 Understand the effect on the probability of ruin, in both finite and infinite time, of changing parameter values by reasoning or simulation

4.1.6 Calculate probabilities of ruin by simulation

4.2 Understand and use run-off triangles to estimate claims (Chapter 19)

4.2.1 Understand what a development factor is and show how a set of assumed development factors can be used to project the future development of a delay triangle

4.2.2 Understand and apply a basic chain ladder method for completing the delay triangle using development factors

4.2.3 Basic chain ladder method and how this can be adjusted to make explicit allowance for inflation

4.2.4 Understand and apply the average cost per claim method for estimating outstanding claim amounts

4.2.5 Understand and apply the Bornhuetter-Ferguson method for estimating outstanding claim amounts

4.2.6 Understand how a statistical model can be used to underpin a run-off triangles approach

4.2.7 Understand the assumptions underlying the application of the methods in 4.2.1 to 4.2.6 above

4.3 Value basic benefit guarantees using simulation techniques

5 Option theory (30%)

The construction and evaluation of common forward and option contracts as well as theoretical models for derivatives and option pricing, in particular the theory and application of binomial and Black-Scholes models.

5.1 Understand the principles of option pricing and valuations (Chapters 10 – 15)

5.1.1 What is meant by arbitrage and a complete market

5.1.2 Factors that affect option prices 5.1.3 Determine specific results for options that are not model dependent:

- Show how to value a forward contract
- Develop upper and lower bounds for European and American call and put options

5.1.4 The meaning of put-call parity

5.2 Understand the principles of the binomial option-pricing model and its application

5.2.1 Use binomial trees and lattices to value options and solve simple examples

5.2.2 Determine the risk-neutral pricing measure for a binomial lattice and describe the risk-neutral pricing approach to the pricing of equity options

5.2.3 Difference between the real-world measure and the risk-neutral measure and why the risk-neutral pricing approach is seen as a computational tool (rather than a realistic representation of price dynamics in the real world)

5.2.4 The alternative names for the risk-neutral and state-price deflator approaches to pricing

5.2.5 Apply the state-price deflator approach to the binomial model and understand its equivalence to the risk-neutral pricing approach What is meant by risk-neutral pricing and the equivalent martingales measure

5.2.6 Use the martingale approach to pricing and hedging using the binomial model

5.3 Understand the principles of the Black-Scholes derivative-pricing model and its application

5.3.1 Underlying principles of the Black-Scholes partial differential equation both in its basic and Garman-Kohlhagen forms

5.3.2 Use the Black-Scholes model to price and hedge a simple derivative contract using the martingale approach

5.3.3 Value options and solve simple examples using the Black-Scholes model

5.3.4 Apply the state-price deflator approach to the Black-Scholes model and understand its equivalence to the risk-neutral pricing approach

5.3.5 Validity of the assumptions underlying the Black-Scholes model

5.3.6 Commonly used terminology for the first and, where appropriate, second partial derivatives (the Greeks) of an option price

2 Changes to the Core Reading

The Core Reading related to Chapter 3 (Stochastic dominance and behavioural finance) and Chapter 5 (Stochastic models of investment returns), has been removed from the course. As such these two chapters have been removed from the course and all the other chapters have been renumbered accordingly.

Much of the Core Reading surrounding the Ornstein-Uhlenbeck process and stochastic models of the term structure of interest rates has been removed. The text has remained in the course, as ActEd text. So:

- following the heading "The Ornstein-Uhlenbeck process" on page 22 of Chapter 10, there is no Core Reading in the remainder of the chapter (now Chapter 8).
- In Sections 3.2 to 3.4 of Chapter 18 (now Chapter 16), much of the Core Reading has been removed.

For the avoidance of doubt, we provide the following replacement pages:

- Pages 22 to 27 of the new Chapter 8.
- Pages 16 to 26 of the new Chapter 16.

3 Changes to the ActEd material

Chapter 3 and Chapter 5 have been removed as explained above in the “Changes to Core Reading”.

As explained above in the “Changes to Core Reading” section, some of the text relating to the Ornstein-Uhlenbeck process and stochastic models of the term structure of interest rates has been changed from bold (Core Reading text) to non-bold (ActEd text). Replacement pages are provided as explained above.

4 Changes to the X Assignments

All of the X Assignments have undergone a reallocation of marks and this has involved some questions being rewritten slightly and some parts being removed.

Because some Core Reading has been removed, assignments X1 and X2 have been reshuffled with some new questions added.

The 2024 X Assignments are provided in the replacement pages.

5 Changes to the Y Assignments

There have been no ***non-trivial*** changes to the Y Assignments.

6 Other tuition services

In addition to the CMP you might find the following services helpful with your study.

6.1 Study material

We also offer the following study material in Subject CM2:

- Flashcards
- Revision Notes
- ASET (ActEd Solutions with Exam Technique) and Mini-ASET
- Mock Exam and AMP (Additional Mock Pack).

For further details on ActEd's study materials, please refer to the 2024 *Student Brochure*, which is available from the ActEd website at **ActEd.co.uk**.

6.2 Tutorials

We offer the following (face-to-face and/or online) tutorials in Subject CM2:

- a set of Regular Tutorials (lasting four full days)
- a Block (or Split Block) Tutorial (lasting four full days)
- an Online Classroom.

For further details on ActEd's tutorials, please refer to our latest *Tuition Bulletin*, which is available from the ActEd website at **ActEd.co.uk**.

6.3 Marking

You can have your attempts at any of our assignments or mock exams marked by ActEd. When marking your scripts, we aim to provide specific advice to improve your chances of success in the exam and to return your scripts as quickly as possible.

For further details on ActEd's marking services, please refer to the 2024 *Student Brochure*, which is available from the ActEd website at **ActEd.co.uk**.

6.4 Feedback on the study material

ActEd is always pleased to receive feedback from students about any aspect of our study programmes. Please let us know if you have any specific comments or general suggestions about how we can improve the study material. We will incorporate as many of your suggestions as we can when we update the course material each year.

If you have any comments on this course, please send them by email to **CM2@bpp.com**.

Then:

$$\ln S_T = \ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)T + \sigma W_T$$

Since $E[W_T]=0$, $Var(W_T)=T$ and all other terms are deterministic we have:

$$E[\ln S_T] = E\left[\ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)T\right] + E[\sigma W_T]$$

$$= \ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)T + 0$$

and:

$$Var(\ln S_T) = Var\left(\ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right)T\right) + Var(\sigma W_T) = 0 + \sigma^2 T$$

as required.

In conclusion, the distribution of a geometric Brownian motion at a time *T* is lognormally distributed:

$$\ln(S_T) \sim N\left(\ln(S_0) + \left(\mu - \tfrac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$$

The Ornstein-Uhlenbeck process

Consider applying the function $f(x,t) = xe^{\gamma t}$ to the Ito diffusion defined by:

$$X_T = X_0 - \int_0^T \gamma X_t dt + \int_0^T \sigma dW_t \quad \Leftrightarrow \quad dX_t = -\gamma X_t dt + \sigma dW_t$$

This is known as the Ornstein-Uhlenbeck process.

We need:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(xe^{\gamma t}\right) = \gamma xe^{\gamma t} = \gamma f(x,t)$$

$$\frac{\partial f(x,t)}{\partial x} = \frac{\partial}{\partial x}\left(xe^{\gamma t}\right) = e^{\gamma t}$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left(e^{\gamma t}\right) = 0$$

so:

$$df(X_t,t) = \gamma X_t e^{\gamma t} dt + e^{\gamma t} dX_t$$

$$= \sigma e^{\gamma t} dW_t$$

When applying Ito's Lemma we use $\mu(X_t,t) = -\gamma X_t$ and $\sigma(X_t,t) = \sigma$, therefore we have:

$$df(X_t,t) = \left(\frac{\partial f}{\partial t} + \mu(X_t,t)\frac{\partial f}{\partial X_t} + \frac{1}{2}\sigma^2(X_t,t)\frac{\partial^2 f}{\partial X_t^2}\right)dt + \sigma(X_t,t)\frac{\partial f}{\partial X_t}dW_t$$

$$= \left(\gamma X_t e^{\gamma t} - \gamma X_t e^{\gamma t} + \frac{1}{2}\sigma^2 \times 0\right)dt + \sigma e^{\gamma t}dW_t$$

$$= \sigma e^{\gamma t}dW_t$$

As a stochastic integral equation we have:

$$\int_0^T df(X_t,t) = \int_0^T \sigma e^{\gamma t}dW_t$$

$$\Rightarrow \quad f(X_T,T) - f(X_0,0) = \int_0^T \sigma e^{\gamma t}dW_t$$

$$\Rightarrow \quad X_T e^{\gamma T} = X_0 + \sigma\int_0^T e^{\gamma t}dW_t$$

$$\Rightarrow \quad X_T = X_0 e^{-\gamma T} + \sigma\int_0^T e^{-\gamma(T-t)}dW_t$$

Question

The stochastic differential equation $dX_t = -\gamma X_t dt + \sigma dW_t$ can also be solved using the integrating factor $e^{\gamma t}$.

Show that you get the same solution if you solve the SDE using an integrating factor.

Solution

We want to solve the equation:

$$dX_t = -\gamma X_t dt + \sigma dW_t$$

or $\quad dX_t + \gamma X_t dt = \sigma dW_t$

Multiplying through by the integrating factor $e^{\gamma t}$ and then changing the dummy variable to s gives:

$$e^{\gamma s}dX_s + \gamma e^{\gamma s}X_s ds = \sigma e^{\gamma s}dW_s$$

The left-hand side is now the differential of a product. So we have:

$$d\left(e^{\gamma s}X_s\right)=\frac{\partial}{\partial s}\left(e^{\gamma s}X_s\right)ds+\frac{1}{2}\frac{\partial^2}{\partial s^2}\left(e^{\gamma s}X_s\right)(ds)^2+\ldots$$

$$+\frac{\partial}{\partial X_s}\left(e^{\gamma s}X_s\right)dX_s+\frac{1}{2}\frac{\partial^2}{\partial X_s^2}\left(e^{\gamma s}X_s\right)(dX_s)^2+\ldots$$

$$+\frac{\partial^2}{\partial s\partial X_s}\left(e^{\gamma s}X_s\right)dsdX_s+\ldots$$

$$=\gamma e^{\gamma s}X_s ds+(e^{\gamma s}+\gamma e^{\gamma s}ds)dX_s$$

$$=\gamma e^{\gamma s}X_s ds+(e^{\gamma s}+\gamma e^{\gamma s}ds)(-\gamma X_s ds+\sigma dW_s)$$

$$=\sigma e^{\gamma s}dW_s$$

Now we can integrate between 0 and *t* to get:

$$e^{\gamma t}X_t-e^{\gamma 0}X_0=\sigma\int_0^t e^{\gamma s}dW_s$$

Finally, we can rearrange this to get the desired form:

$$X_t=X_0e^{-\gamma t}+\sigma\int_0^t e^{-\gamma(t-s)}dW_s$$

More generally we have:

$$X_T=X_te^{-\gamma(T-t)}+\sigma\int_t^T e^{-\gamma(T-u)}dW_u$$

On this basis we can examine the distributional properties of X_T. Note the Ito integral has a deterministic integrand and so:

$$E\left[X_T\right]=E\left[X_te^{-\gamma(T-t)}+\sigma\int_t^T e^{-\gamma(T-u)}dW_u\right]$$

$$=X_te^{-\gamma(T-t)}$$

This is because the expectation of an Ito integral is zero, *ie*:

$$E\left[\sigma\int_t^T e^{-\gamma(T-u)}dW_u\right]=0$$

And:

$$Var(X_T) = E\left[\left(X_T - E[X_T]\right)^2\right]$$

$$= Var\left(X_t e^{-\gamma(T-t)} + \sigma \int_t^T e^{-\gamma(T-u)} dW_u\right)$$

$$= \underbrace{Var\left(X_t e^{-\gamma(T-t)}\right)}_{=0} + \sigma^2 Var\left(\int_t^T e^{-\gamma(T-u)} dW_u\right) + \underbrace{2Cov\left(X_t e^{-\gamma(T-t)}, \sigma \int_t^T e^{-\gamma(T-u)} dW_u\right)}_{=0}$$

By Ito isometry:

$$Var(X_T) = \sigma^2 \int_t^T e^{-2\gamma(T-u)} du$$

$$= \frac{\sigma^2}{2\gamma}\left[e^{-2\gamma(T-u)}\right]_t^T$$

$$= \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma(T-t)}\right)$$

For large $(T-t)$ this is approximately $\frac{\sigma^2}{2\gamma}$ while for small $(T-t)$ it is (unsurprisingly) close to zero.

Question

Why is this unsurprising?

Solution

$T-t$ is the length of time over which the process is being observed. When this quantity is small it means that there's little opportunity for the process value X_T to deviate very far from X_t. This behaviour is captured by having a variance close to zero.

The mean-reverting process

The mean-reverting process, defined by the SDE:

$$dY_t = \gamma(\mu - Y_t)dt + \sigma dW_t$$

is based on the Ornstein-Uhlenbeck process. In the mean-reverting process, the process is pulled back to some equilibrium level, μ, at a rate determined by $\gamma > 0$. Note that this process can go negative.

We can investigate this by considering $e^{\gamma t}Y_t$ (noting that the equation $dx = \gamma x dt$ implies the solution $x = e^{\gamma t}$) then:

$$d\left(e^{\gamma t}Y_t\right) = \frac{\partial}{\partial t}\left(e^{\gamma t}Y_t\right)dt + \frac{1}{2}\frac{\partial^2}{\partial t^2}\left(e^{\gamma t}Y_t\right)(dt)^2 + \ldots$$

$$+\frac{\partial}{\partial Y_t}\left(e^{\gamma t}Y_t\right)dY_t + \frac{1}{2}\frac{\partial^2}{\partial Y_t^2}\left(e^{\gamma t}Y_t\right)(dY_t)^2 + \ldots$$

$$+\frac{\partial^2}{\partial t \partial Y_t}\left(e^{\gamma t}Y_t\right)dtdY_t + \ldots$$

$$= \gamma e^{\gamma t}Y_t dt + (e^{\gamma t} + \gamma e^{\gamma t}dt)dY_t$$

$$= \gamma e^{\gamma t}Y_t dt + (e^{\gamma t} + \gamma e^{\gamma t}dt)(\gamma(\mu - Y_t)dt + \sigma dW_t)$$

$$= \gamma\mu e^{\gamma t}dt + \sigma e^{\gamma t}dW$$

so:

$$d\left(e^{\gamma t}Y_t\right) = \gamma\mu e^{\gamma t}dt + \sigma e^{\gamma t}dW_t$$

Changing the variable of integration and integrating both sides between times *t* and *T* gives:

$$\int_t^T d\left(e^{\gamma s}Y_s\right) = \int_t^T \gamma\mu e^{\gamma s}ds + \int_t^T \sigma e^{\gamma s}dW_s$$

which implies that:

$$e^{\gamma T}Y_T = e^{\gamma t}Y_t + \int_t^T \gamma\mu e^{\gamma s}ds + \int_t^T \sigma e^{\gamma s}dW_s$$

because there are no random variables in the integrands, these are straightforward, and:

$$Y_T = e^{-\gamma(T-t)}Y_t + \mu\left(1 - e^{-\gamma(T-t)}\right) + \sigma\int_t^T e^{-\gamma(T-s)}dW_s$$

This result implies that:

$$E\left[Y_T\right] = e^{-\gamma(T-t)}Y_t + \mu\left(1 - e^{-\gamma(T-t)}\right)$$

while

$$Var\left(Y_T\right) = E\left[\left(\sigma\int_t^T e^{-\gamma(T-s)}dW_s\right)^2\right]$$

By Ito isometry:

$$Var\left(Y_T\right)=\frac{\sigma^2}{2\gamma}\left(1-e^{-2\gamma(T-t)}\right)$$

Observe that this tells us that:

$$Y_T \sim N\left(e^{-\gamma(T-t)}Y_t+\mu\left(1-e^{-\gamma(T-t)}\right),\frac{\sigma^2}{2\gamma}\left(1-e^{-2\gamma(T-t)}\right)\right)$$

This process is used again in the chapter on the term structure of interest rates.

Solution

Martingale processes have zero drift. In order for $\frac{P(t,T)}{B_t}$ to have zero drift the coefficient of the dt term in the stochastic differential equation must be zero.

3.2 The Vasicek model (1977)

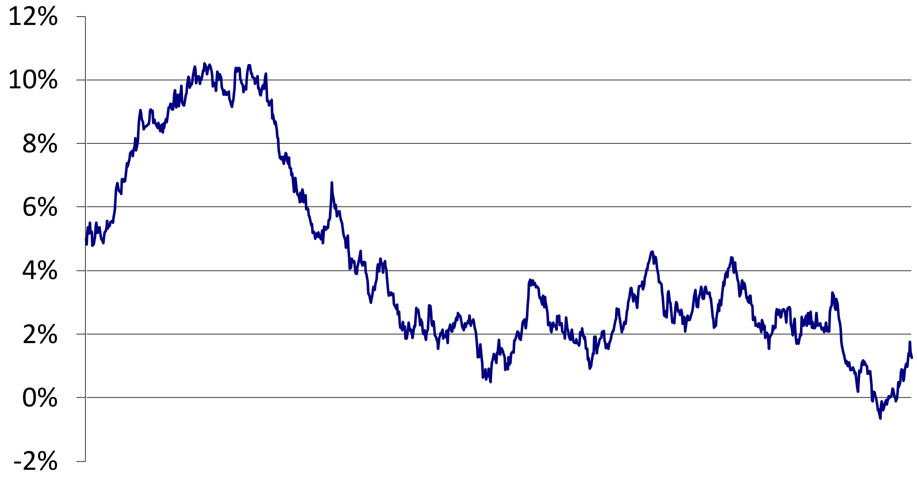
Vasicek assumes that:

$$dr_t = \alpha(\mu - r_t)dt + \sigma d\tilde{W}_t$$

for positive constants α , μ and, σ .

Here μ represents the 'mean' level of the short rate. If the short rate grows (driven by the stochastic term) the drift becomes negative, pulling the rate back to μ . The speed of the 'reversion' is determined by α . If α is high, the reversion will be very quick.

The graph below show a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.02$.



Example simulation of short rate from the Vasicek model

The derivation of the bond pricing formula is not a course requirement, however, we show it here for completeness.

The above SDE yields the partial differential equation:

$$\frac{\partial g(t,r_t)}{\partial t} + \frac{\partial g(t,r_t)}{\partial r_t}\alpha(\mu - r_t) + \frac{1}{2}\frac{\partial^2 g(t,r_t)}{\partial r_t^2}\sigma^2 - r_t g(t,r_t) = 0$$

by making the substitution $\mu(t,r_t) = \alpha(\mu - r_t)$ and letting $\sigma(t,r_t)$ be constant.

To establish the form of $g(t,r_t)=P(t,T)$, recall that for the Ornstein-Uhlenbeck process:

$$r_t=\mu+(r_0-\mu)e^{-\alpha t}+\sigma\int_0^t e^{-\alpha(t-u)}d\tilde{W}_u$$

and so, integrating again:

$$\int_0^T r_s ds=\mu T+\frac{1}{\alpha}(r_0-\mu)\left(1-e^{-\alpha T}\right)+\frac{\sigma}{\alpha}\int_0^T\left(1-e^{-\alpha(T-u)}\right)d\tilde{W}_u$$

This implies that $\int_0^T r_s ds$ is normally distributed with conditional mean:

$$E\left[\int_0^T r_s ds\right]=\mu T+\frac{1}{\alpha}(r_0-\mu)\left(1-e^{-\alpha T}\right)$$

Using Ito isometry:

$$Var_Q\left[\int_0^T \nu(X_t,t)dW_t\right]=\int_0^T E\left[\nu^2(X_t,t)\right]dt$$

we can see that:

$$Var_Q\left[\int_0^T r_s ds\right]=\frac{\sigma^2}{\alpha^2}\left(T-\frac{2}{\alpha}(1-e^{-\alpha T})+\frac{1}{2\alpha}(1-e^{-2\alpha T})\right)$$

Since $\int_0^T r_s ds$ is normally distributed, using the moment generating function of a normal, $N\left(\eta,\nu^2\right)$, random variable:

$$E\left[e^{tX}\right]=e^{\eta t+\frac{1}{2}\nu^2t^2}$$

by putting $t=-1$, we can say that:

$$P(0,T)=E_Q\left[\exp\left(-\int_0^T r_s ds\right)\right]$$

$$=\exp\left(-\left(\mu T+\frac{1}{\alpha}(r_0-\mu)\left(1-e^{-\alpha T}\right)\right)+\frac{1}{2}\left(\frac{\sigma^2}{\alpha^2}\left(T-\frac{2}{\alpha}(1-e^{-\alpha T})+\frac{1}{2\alpha}(1-e^{-2\alpha T})\right)\right)\right)$$

In general, by setting:

$$b(t,T)=\frac{1}{\alpha}\left(1-e^{-\alpha(T-t)}\right)$$

and:

$$a(t,T)=\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)\left(b(t,T)-T+t\right)-\frac{\sigma^2}{4\alpha}b^2(t,T)$$

we have:

$$P(t,T)=e^{a(t,T)-b(t,T)r_t}$$

By defining $\tau=T-t$, then equivalently we have:

$$P(t,T)=e^{a(\tau)-b(\tau)r_t}$$

where:

$$b(\tau)=\frac{1}{\alpha}\left(1-e^{-\alpha\tau}\right)$$

and:

$$a(\tau)=\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)\left(b(\tau)-\tau\right)-\frac{\sigma^2}{4\alpha}b^2(\tau)$$

Question

Show that the instantaneous forward rate for the Vasicek model can be expressed as:

$$f(t,T)=r(t)e^{-\alpha\tau}+\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)(1-e^{-\alpha\tau})+\frac{\sigma^2}{2\alpha^2}(1-e^{-\alpha\tau})e^{-\alpha\tau}$$

where $\tau=T-t$.

Solution

We have a formula for $P(t,T)$ under this model. So the instantaneous forward rate can be derived from this using the relationship:

$$f(t,T)=-\frac{\partial}{\partial T}\log P(t,T)$$

By use of the chain rule, and noting that $\frac{\partial \tau}{\partial T}=1$, this gives:

$$f(t,T)=-\frac{\partial}{\partial T}[a(\tau)-b(\tau)r(t)]$$

$$=-\frac{\partial \tau}{\partial \tau}\frac{\partial}{\partial T}[a(\tau)-b(\tau)r(t)]$$

$$=-\frac{\partial \tau}{\partial T}\frac{\partial}{\partial \tau}[a(\tau)-b(\tau)r(t)]=-a'(\tau)+b'(\tau)r(t)$$

From the definitions of $b(\tau)$ and $a(\tau)$, we find that:

$$b'(\tau)=\frac{d}{d\tau}\left(\frac{1-e^{-\alpha\tau}}{\alpha}\right)=e^{-\alpha\tau}$$

and:

$$a'(\tau)=\frac{d}{d\tau}\left[(b(\tau)-\tau)\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)-\frac{\sigma^2}{4\alpha}b(\tau)^2\right]$$

$$=(b'(\tau)-1)\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)-\frac{\sigma^2}{4\alpha}\times 2b(\tau)b'(\tau)$$

$$=(e^{-\alpha\tau}-1)\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)-\frac{\sigma^2}{2\alpha}\left(\frac{1-e^{-\alpha\tau}}{\alpha}\right)e^{-\alpha\tau}$$

$$=-(1-e^{-\alpha\tau})\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)-\frac{\sigma^2}{2\alpha^2}(1-e^{-\alpha\tau})e^{-\alpha\tau}$$

Substituting these expressions into the general formula for $f(t,T)$ gives the required answer.

Question

Write down an expression in terms of the model parameters for the long rate, *ie* the instantaneous forward rate corresponding to $T-t=\infty$, according to the Vasicek model.

Solution

Letting $T\to\infty$ (and hence $\tau\to\infty$) in the equation for $f(t,T)$ gives:

$$f(t,\infty)=r(t)\times 0+\left(\mu-\frac{\sigma^2}{2\alpha^2}\right)(1-0)+\frac{\sigma^2}{2\alpha^2}(1-0)\times 0=\mu-\frac{\sigma^2}{2\alpha^2}$$

The curves shown on the graph of gilt yields in Section 1 were fitted using a Vasicek model with parameter values $\alpha = 0.131$, $\mu = 0.083$ and $\sigma = 0.037$.

Question

'The particular model used for the graph implies that interest rates are mean-reverting to the value $\mu = 0.083$.'

True or false?

Solution

The dynamics of $r(t)$ for this particular Vasicek model are:

$$dr(t) = -0.131\,[r(t) - 0.083]dt + 0.037 d\tilde{W}(t)$$

under the *risk-neutral* probability measure Q. Under this measure $\tilde{W}(t)$ is standard Brownian motion and therefore has zero drift and the process mean-reverts to the value 0.083.

However, under the *real-world* probability measure P, $\tilde{W}(t)$ would have non-zero drift and the process will mean-revert to a different value. In fact, although we will not prove it here, the long-term rate in the real world can be found from the formula derived in the previous question, namely:

$$\mu - \frac{\sigma^2}{2\alpha^2} = 0.0431 \quad ie \ \ 4.31\%$$

3.3 The Cox-Ingersoll-Ross (CIR) model (1985)

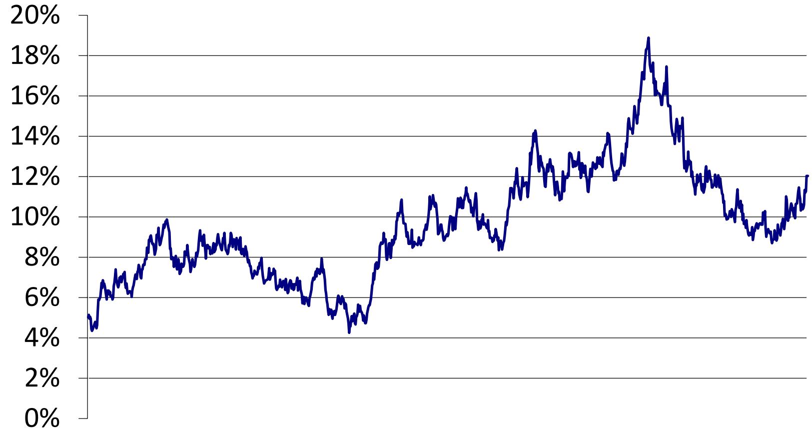
In Vasicek's model (and Hull-White, below) interest rates are not strictly positive. This assumption is not ideal for a short-rate model. CIR use the Feller, or square root mean reverting process, which is positive (it can instantaneously touch 0 but immediately rebounds):

$$dr_t = \alpha(\mu - r_t)dt + \sigma\sqrt{r_t}\,d\tilde{W}_t$$

for constants $\alpha > 0$, $\mu > 0$ and, σ.

We can see that the form of the drift of r_t is the same as for the Vasicek model. The critical difference between the two models occurs in the volatility, which is increasing in line with the square root of r_t. Since this diminishes to zero as r_t approaches zero, and provided σ^2 is not too large ($\sigma^2 \le 2\alpha\mu$), we can guarantee that r_t will not hit zero. Consequently all other interest rates will also remain strictly positive.

The graph below shows a simulation of this process based on the parameter values $\alpha = 0.1$, $\mu = 0.06$ and $\sigma = 0.1$.



Simulation from Cox-Ingersoll-Ross model

It is not possible to solve the SDE for the CIR model.

The associated PDE is:

$$\frac{\partial g(t,r_t)}{\partial t} + \frac{\partial g(t,r_t)}{\partial r_t}\alpha(\mu - r_t) + \frac{1}{2}\sigma^2 r_t \frac{\partial^2 g(t,r_t)}{\partial r_t^2} - r_t\, g(t,r_t) = 0$$

and, again, $P(t,T) = e^{a(\tau) - b(\tau) r_t}$ with:

$$b(\tau) = \frac{2\left(e^{\gamma\tau} - 1\right)}{(\gamma + \alpha)\left(e^{\gamma\tau} - 1\right) + 2\gamma}$$

$$a(\tau) = \frac{2\alpha\mu}{\sigma^2}\ln\left(\frac{2\gamma e^{\frac{1}{2}(\gamma+\alpha)\tau}}{(\gamma + \alpha)\left(e^{\gamma\tau} - 1\right) + 2\gamma}\right)$$

where $\gamma = \sqrt{\alpha^2 + 2\sigma^2}$.

It turns out that these values for a and b are not that different from those in Vasicek's model.

The distribution of r_t is given by a 'non-central chi-squared' distribution. this is a 'fat-tailed' distribution.

If $X_1, X_2, \ldots, X_n$ are independent random variables, each with a $N(0,1)$ distribution, then $Y = X_1^2 + X_2^2 + \cdots + X_n^2$ has a chi-square distribution with n degrees of freedom.

If $X_1, X_2, \ldots, X_n$ are independent random variables and $X_i \sim N(d_i, 1)$, then $Y_d = X_1^2 + X_2^2 + \cdots + X_n^2$ is said to have a *non-central* chi-squared distribution with n degrees of freedom and non-centrality parameter $d = \sum_{i=1}^{n} d_i^2$.

So the non-central chi-squared distribution can be thought of as a lopsided version of the ordinary chi-square distribution.

Question

What is the mean of the non-central chi-squared distribution with n degrees of freedom and non-centrality parameter d?

Solution

Since $X_i \sim N(d_i, 1)$, we find that:

$$E[X_i^2] = Var(X_i) + [E(X_i)]^2 = 1 + d_i^2$$

It follows that:

$$E[Y_d] = E[X_1^2 + X_2^2 + \cdots + X_n^2] = \sum_{i=1}^{n}(1 + d_i^2) = n + d$$

3.4 Vasicek and CIR yield curves

Recall:

$$\boldsymbol{r(t,T) = -\frac{\ln P(t,T)}{T-t}}$$

and:

$$P(t,T) = e^{a(t,T) - b(t,T) r_t}$$

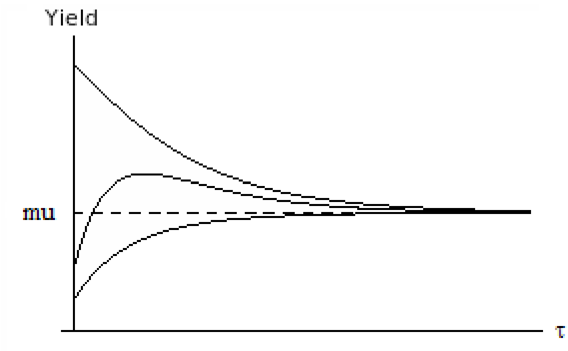
so we have:

$$\boldsymbol{r(t,T) = -\frac{a(t,T) - b(t,T) r_t}{T-t}}$$

Alternatively:

$$r(\tau) = -\frac{a(\tau) - b(\tau) r_t}{\tau}$$

Then the yield curves coming out of the Vasicek model are of three (related) types:



These yield curves are generally described as being:

- normal, with short-term yields being lower than long-term yields
- inverted, with short-term yields being higher than long-term yields
- humped, *ie* a yield curve with a turning point.

Recall that the duration, *D*, of an asset, whose interest rate dependent price is given by *B*, is defined by:

$$\frac{\Delta B}{B} = -D\delta y$$

where *y* is the instrument's yield. In the context of interest rate models, this is equivalent to:

$$\frac{\partial P(t,T)}{\partial r_t} = -DP(t,T) = -b(t,T)P(t,T)$$

and there is a connection between this duration D, and the function *b*.

The relationship above comes from the fact that:

$$\frac{\Delta B}{B} = -D\delta y \quad \Leftrightarrow \quad \frac{\Delta B}{\delta y} = -BD$$

and $\frac{\Delta B}{\delta y}$ is the rate at which the bond price changes with respect to a change in its yield,

ie $\frac{\partial P(t,T)}{\partial r_t}$.

As $P(t,T) = e^{a(t,T)-b(t,T)r_t}$, then:

$$\frac{\partial P(t,T)}{\partial r_t} = \frac{\partial}{\partial r_t}\left(e^{a(t,T)-b(t,T)r_t}\right) = -b(t,T)e^{a(t,T)-b(t,T)r_t} = -b(t,T)P(t,T)$$

3.5 The Hull-White model (1990)

The Hull-White model is an extension of Vasicek where the mean-reversion level, μ, is a deterministic function of time:

$$dr_t = \alpha(\mu(t) - r_t)dt + \sigma d\tilde{W}_t$$

for constants $\alpha > 0$ and σ.

In some representations, the parameter α is also allowed to be a function of time. This is known as the *extended Vasicek model*.

This yields a PDE similar to Vasicek, and so we can start by 'guessing' that $P(t,T) = e^{a(t,T)-b(t,T)r_t} = g(t,r_t)$ and so:

$$\frac{\partial g(t,r_t)}{\partial t} + \frac{\partial g(t,r_t)}{\partial r_t}\alpha(\mu(t)-r_t) + \frac{1}{2}\frac{\partial^2 g(t,r_t)}{\partial r_t^2}\sigma^2 - r_t g(t,r_t) = 0$$

By noting that:

$$\frac{\partial g(t,r_t)}{\partial t} = g(t,r_t)\left(\frac{\partial a(t,T)}{\partial t} - \frac{\partial b(t,T)}{\partial t}r_t\right)$$

$$\frac{\partial g(t,r_t)}{\partial r_t} = -g(t,r_t)b(t,T)$$

$$\frac{\partial^2 g(t,r_t)}{\partial r_t^2} = g(t,r_t)b^2(t,T)$$

Then the Hull-White PDE becomes:

$$g(t,r_t)\left(\frac{\partial a(t,T)}{\partial t} - \frac{\partial b(t,T)}{\partial t}r_t\right) - \alpha g(t,r_t)b(t,T)(\mu(t)-r_t) + \frac{1}{2}g(t,r_t)b^2(t,T)\sigma^2 - r_t g(t,r_t) = 0$$

$$\Rightarrow \quad g(t,r_t)\left(\frac{\partial a(t,T)}{\partial t} - \frac{\partial b(t,T)}{\partial t}r_t - \alpha b(t,T)(\mu(t)-r_t) + \frac{1}{2}b^2(t,T)\sigma^2 - r_t\right) = 0$$

$$\Rightarrow \quad g(t,r_t)\left(r_t\left(\alpha b(t,T) - \frac{\partial b(t,T)}{\partial t} - 1\right) + \frac{\partial a(t,T)}{\partial t} - \alpha b(t,T)\mu(t) + \frac{1}{2}b^2(t,T)\sigma^2\right) = 0$$

By letting $a'(t,T) = \frac{\partial a(t,T)}{\partial t}$ and $b'(t,T) = \frac{\partial b(t,T)}{\partial t}$ then we have:

$$g(t,r_t)\left(r_t\left(\alpha b(t,T) - b'(t,T) - 1\right) + a'(t,T) - \alpha b(t,T)\mu(t) + \frac{1}{2}b^2(t,T)\sigma^2\right) = 0$$

This is essentially Vasicek but we have:

$$b(t,T)=\int_t^T \exp\left(-\int_t^s \alpha(u)du\right)ds$$

As α has been taken as a constant in the model above we have:

$$\begin{aligned}b(t,T)&=\int_t^T \exp\left(-\int_t^s \alpha du\right)ds\\&=\int_t^T \exp\left(-\alpha\left(s-t\right)\right)ds\\&=\frac{1}{\alpha}\left(1-e^{-\alpha(T-t)}\right)\end{aligned}$$

So $b(t,T)$ is the same as for the Vasicek model.

$$a(t,T)=\int_t^T \left(-\alpha\mu(s)b(s,T)+\tfrac{1}{2}\sigma^2 b^2(s,T)\right)ds$$

Question

Show that $a(t,T)$ and $b(t,T)$ satisfy the equation

$$g(t,r_t)\left(r_t\left(\alpha b(t,T)-b'(t,T)-1\right)+a'(t,T)-\alpha b(t,T)\mu(t)+\frac{1}{2}b^2(t,T)\sigma^2\right)=0$$

Solution

By differentiating $a(t,T)$ and $b(t,T)$ we have:

$$a'(t,T)=\frac{\partial a(t,T)}{\partial t}=\alpha\mu(t)b(t,T)-\tfrac{1}{2}\sigma^2 b^2(t,T)$$

and

$$b'(t,T)=\frac{\partial b(t,T)}{\partial t}=-e^{-\alpha(T-t)}$$

Substituting these into the given equation gives:

$$g(t,r_t)\Big(r_t\left(\alpha b(t,T)-b'(t,T)-1\right)+a'(t,T)-\alpha b(t,T)\mu(t)+\tfrac{1}{2}b^2(t,T)\sigma^2\Big)$$

$$=g(t,r_t)\begin{pmatrix} r_t\left(\alpha b(t,T)+e^{-\alpha(T-t)}-1\right)+\alpha\mu(t)b(t,T) \\ -\tfrac{1}{2}\sigma^2 b^2(t,T)-\alpha b(t,T)\mu(t)+\tfrac{1}{2}b^2(t,T)\sigma^2 \end{pmatrix}$$

$$=g(t,r_t)\left(r_t\times 0+0\right)$$

$$=0$$

as required.

The advantage of this model over Vasicek is that $\mu(t)$ can be chosen to reproduce (as closely as possible) the exact yield curve, rather than the restricted forms of the Vasicek model.

4 Summary of short-rate modelling

4.1 Summary of models

The properties of these models are summarised below:

Model	Dynamics	$r_t > 0$ for all t	Distribution of r_t
Vasicek	$dr_t = \alpha(\mu - r_t)dt + \sigma d\tilde{W}_t$	No	Normal
CIR	$dr_t = \alpha(\mu - r_t)dt + \sigma\sqrt{r_t}\,d\tilde{W}_t$	Yes	Non-central chi-squared
Hull-White	$dr_t = \alpha(\mu(t) - r_t)dt + \sigma d\tilde{W}_t$	No	Normal

4.2 Limitations

Question

What is a one-factor term structure model?

Solution

A one-factor model is a model in which interest rates are assumed to be influenced by a single source of randomness.

Bearing in mind that the purpose of interest rate models is to price interest rate derivatives, there are some short-comings of short-rate models:

- **Single factor short-rate models mean that all maturities behave in the same way - there is no independence.**
- **There is little consistency in valuation between the models.**
- **They are difficult to calibrate.**

One-factor models, such as Vasicek and CIR, have certain limitations with which it is important to be familiar.

X1.1 (i) *Technical analysis* is the study of chart patterns of various asset prices. In your own words, explain whether this can be used to an investor's advantage if the Efficient Markets Hypothesis (EMH) holds. [2]

(ii) *Insider trading* is illegal in the UK stock market. In your own words, explain what this suggests about the EMH. [2]

(iii) *Fundamental analysis* includes the analysis of balance sheets, consideration of company strategy, the environment in which the company operates *etc*. In your own words, explain how this relates to the EMH. [2]

[Total 6]

X1.2 (i) An investor has the utility function $U(w) = -\exp\left(-\frac{w}{100}\right)$.

Determine whether the investor exhibits increasing, constant or decreasing absolute and relative risk aversion. [4]

(ii) The investor has an initial wealth of $1,000$ and is offered a gamble with a payoff described by a random variable:

$$X = \begin{cases} +100 & \text{with probability } 0.5 \\ -50 & \text{with probability } 0.5 \end{cases}$$

Calculate the investor's certainty equivalent of this gamble. [3]

[Total 7]

X1.3 (i) You are given the choice of only two assets, A and B. The expected returns and variances of return of the two assets are:

$$E_A = 13\% \qquad E_B = 5\%$$

$$V_A = 36\%\% \qquad V_B = 4\%\%$$

Determine the equation of the efficient frontier in (E, σ) space in the special case where the returns on Assets A and B are perfectly correlated. Comment on your result. [4]

(ii) State, giving the relevant equations, how your approach in (ii) would be modified if there were more than two assets? [*Numerical calculations are not required*.] [5]

(iii) Explain why an investor may choose not to use variance as the measure of risk in assessing potential investment in assets A and B. [2]

[Total 11]

X1.4 An investor is trying to choose between the investments whose distributions of returns are described below:

Investment A:	0.4 probability that it will return 10%
	0.2 probability that it will return 15%
	0.4 probability that it will return 20%
Investment B:	0.25 probability that it will return 10%
	0.70 probability that it will return 15%
	0.05 probability that it will return 40%
Investment C:	A uniform distribution on the range 10% to 20%

Calculate the following for each investment:

(i) expected return [2]

(ii) variance of return [4]

(iii) semi-variance [4]

(iv) expected shortfall below 12% [4]

(v) shortfall probability below 15%. [1]

[Total 15]

X1.5 (i) State the general form of the equation used in multifactor models of security returns, defining any terms you use. [2]

(ii) In your own words, describe the different categories of factors that are used in these models and illustrate your answer with suitable examples. [7]

[Total 9]

X1.6 Assets A and B have the following distributions of returns in various states:

State	Asset A	Asset B	Probability
1	10%	–12%	0.1
2	8%	0%	0.2
3	6%	3%	0.3
4	4%	16%	0.4

(i) Calculate the correlation coefficient between the returns on asset A and asset B and comment on your answer. [4]

(ii) An investor is going to set up a portfolio consisting entirely of assets A and B. Calculate the proportion of assets that should be invested in asset A to obtain the portfolio with the smallest possible variance. [3]

(iii) Assume that the means and the variances of the returns on assets A and B remain unchanged, but that the correlation ρ_{AB} between assets A and B does change. The investor decides to hold 80% of their wealth in asset A and 20% in asset B. Calculate the range of values of ρ_{AB} such that the portfolio has a smaller variance than if they were to invest everything in asset A. [4]

(iv) Describe the effect on the efficient frontier of introducing a risk-free asset that can be bought or sold in unlimited quantities. [2]

[Total 13]

X1.7 Consider an investment market in which:

- the risk-free rate of return on Treasury bills is 4% *pa*
- the expected return on the market as a whole is 8% *pa*
- the standard deviation of the return on the market as a whole is 30% *pa*
- the assumptions of the capital asset pricing model (CAPM) hold.

(i) Consider an efficient portfolio Z that consists entirely of Treasury bills and non-dividend-paying shares, there being no other types of investment. If Z yields an expected return of 7% *pa*, determine its beta. [2]

(ii) Calculate the standard deviation of returns for Portfolio Z. [2]

(iii) Split the total standard deviation for Portfolio Z into the components attributable to systematic risk and specific risk. [2]

(iv) Calculate the market value of Portfolio Z assuming that its constituent securities are expected to realise a total sum of $100 in one period from now. [2]

[Total 8]

X1.8 (i) In your own words, explain what the separation theorem implies about optimal investment strategies. [2]

(ii) Explain why an individual investor wouldn't hold the market portfolio as part of their investment portfolio in practice. [2]

You are given the following historical information for a share in Company ABC and for a portfolio of 100 shares.

	Return (% *pa*)	Standard deviation of return (% *pa*)	beta
ABC	8.5	20	0.7
Portfolio	10.5	16	1.1

(iii) Use these results to derive the expected return on the market portfolio and the risk-free rate of return assuming the CAPM applies. [4]

A student has commented that ABC's lower return and higher standard deviation, relative to the 100-share portfolio, contradicts the predictions of the CAPM.

(iv) Discuss the student's comment. [3]

[Total 11]

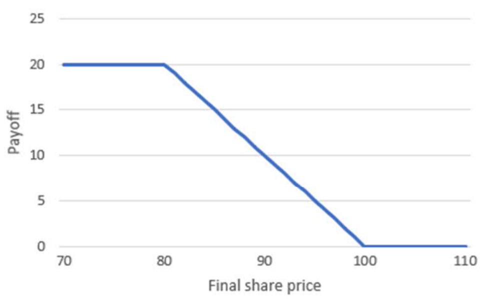
X2.1 You are given the following portfolios:

- Portfolio A – long one call option (strike price 80), short one call option (strike price 100).
- Portfolio B – long one call option (strike price 80), long one put option (strike price 100).
- Portfolio C – short one put option (strike price 80), long one put option (strike price 100).
- Portfolio D – short one call option (strike price 80), long one put option (strike price 100).

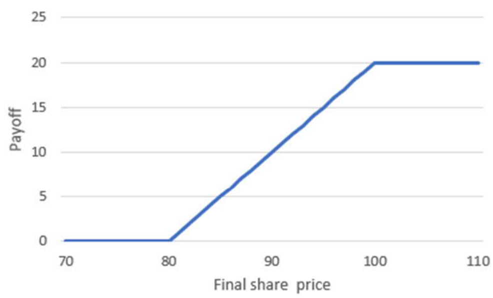
Note that a long position means that the option has been purchased, whereas a short position means that the option has been sold.

State which portfolio corresponds to each of the following four payoff diagrams, which show the payoff of the portfolio at expiry:

(i)

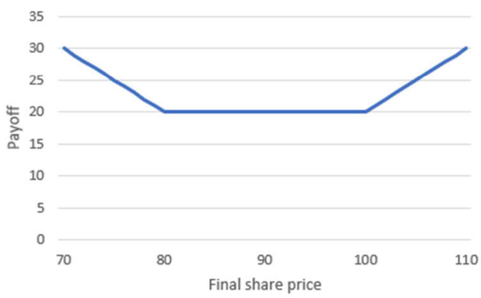


(ii)



(iii)

(iv)



[4]

X2.2 A farmer is due to sell 200 cows at a farmers' market in 6 months' time. The current market value of a cow is $C_0 = 900$ but the farmer is worried about the uncertainty surrounding the price of cows and wishes therefore to enter into a forward contract to sell 200 cows in 6 months for a fixed price F, to be agreed today.

The farmer will receive a one-off payment of 80 per cow in 3 months' time in return for dairy produce. The continuously compounded risk-free rate of interest is $r = 1\%$ *pa*.

Derive the fair value of F, stating any assumptions that you make. [8]

X2.3 (i) Use Taylor's formula to derive Ito's Lemma for a function $f(X_t)$. [2]

An oil trader uses the following model for the short-term behaviour of the oil price X_t, measured in terms of US $100 per barrel:

$$X_t = 0.05\,t + 0.10\,B_t$$

where B_t is a standard Brownian motion.

(ii) Use Ito's Lemma for a function $f(B_t, t)$ to derive the stochastic differential equation for X_t. [4]

A bank offers an exotic derivative whose value is given by:

$$G(X_t) = X_t^2$$

(iii) Use Ito's Lemma to derive the stochastic differential equation for the exotic derivative. [3]
[Total 9]

X2.4 An investment banker wishes to model exchange rate movements between US Dollars and Euros as a geometric Brownian motion. Suppose they decide to use the following stochastic differential equation for this purpose:

$$dX_t = (r_d - r_e)X_t dt + \sigma X_t dB_t$$

where:

- X_t represents the value of US Dollars in terms of Euros
- r_d and r_e are the (constant) short-term interest rates in the US and the Eurozone respectively
- B_t is a standard Brownian motion.

(i) Explain why, in economic terms, the value of the US Dollar is likely to increase when $r_d > r_e$. [2]

(ii) By considering the function $f(X_t) = \log X_t$, use Ito's Lemma to solve the above stochastic differential equation for X_t. [5]

(iii) Let $G_t = \dfrac{1}{X_t}$ denote the value of the Euro in terms of US Dollars. Derive the stochastic differential equation for G_t and comment on your answer. [5]
[Total 12]

X2.5 (i) Write down the probability density function, for an increment over a time lag $t-s$, of general Brownian motion $Z_t = \sigma B_t + \mu t$. [3]

(ii) By first obtaining the stochastic differential equation for the function $f(S_t) = \log S_t$, solve the stochastic differential equation defining geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$ [5]

(iii) S_t, the price of a share at time *t*, is modelled as geometric Brownian motion. If μ = 20% *pa* and σ = 10% *pa*, calculate the probability that the share price will exceed 110 in six months' time given that its current price is 100. [4]

(iv) Calculate a 99% confidence interval for the share price in three months' time given that its current price is 100. [4]

[Total 15]

X2.6 (i) In your own words, describe what is meant by the lognormal model of security prices. [3]

(ii) If X_t is defined to be the deviation of the log of the security price S_t from its trend value, show that changes in X_t over a time interval *h* are stationary. [2]

(iii) Derive expressions for the mean and the variance of the security price S_t. [4]

[Total 9]

X2.7 The stochastic differential equation that implies that the price of an asset at time t, S_t, follows a geometric Brownian motion, is:

$$dS_t = S_t \{\mu\, dt + \sigma\, dZ_t\}$$

where Z_t is a standard Brownian motion process.

Discuss the plausibility of the assumptions behind this equation when it is used as a model of share prices. [7]

X2.8 (i) Explain what the principle of no arbitrage implies about derivative prices. [2]

You are given the following prices of 3-month options, all based on the same (non-dividend-paying) underlying asset with current value $S=100$:

Strike price, K	Option price	
	call, c	put, p
90	15.41	5.19
100	12.03	11.78
110	6.28	15.11

The continuously compounded risk-free rate of interest is $r=1\%$ *pa*.

(ii) Determine whether put-call parity holds for each of the three given strike prices. [3]

(iii) Explain, using a numerical example, how to create an arbitrage opportunity if put-call parity does not hold. [4]

Consider the following portfolio:

- Buy one call option (strike price 90)
- Sell two call options (strike price 100)
- Buy one call option (strike price 110).

(iv) Demonstrate that this portfolio creates an arbitrage profit. [5]

(v) Explain the likely consequences of an arbitrage opportunity existing in real markets. [2]

[Total 16]

X3.1 (i) In the context of a non-dividend-paying security, define the Greeks (in both your own words and using formulae) and state whether each has a positive or negative value for a call option and a put option. [5]

(ii) Consider a 30-day, at-the-money call option on a non-dividend-paying share currently valued at £8. The volatility of the share is 30% and the continuously compounded risk-free rate of interest is 4%.

The outputs of a computer model used to value the option are:

Option value	28.7p	Theta	-0.499p day^{-1}
Delta	0.532	Vega	$0.91\text{p }\%^{-1}$
Gamma	0.00578p^{-1}	Rho	$0.33\text{p }\%^{-1}$

After one day the share price increases by 50p, volatility is reassessed to be 35% *pa* and the risk-free rate moves to 3.5% *pa*. Estimate the new price of the option. [5]

[Total 10]

X3.2 (i) Consider a call option and a put option on a dividend-paying security, each with the same term and exercise price. By considering the put-call parity relationship or otherwise, state the value of n such that:

$$\Delta_c = \Delta_p + n$$

(Δ_c is the delta for the call option and Δ_p is the delta for the put option.) [1]

(ii) Derive similar relationships for the other five Greeks. [5]

(iii) Hence, or otherwise, decide whether or not the following relationship holds:

$$r\rho_c + q\lambda_c + (T-t)\theta_c = r\rho_p + q\lambda_p + (T-t)\theta_p \qquad [2]$$

[Total 8]

X3.3 A non-dividend-paying stock has a current price of £100. In any unit of time the price of the stock is expected to increase by 10% or decrease by 5%. The continuously compounded risk-free interest rate is 4% per unit of time.

A European call option is written with a strike price of £103 and is exercisable after two units of time, at $t = 2$.

Establish, using a binomial tree, the replicating portfolio for the option at the start and end of the first unit of time, *ie* at $t = 0, 1$. Hence, calculate the value of the option at $t = 0$. [11]

X3.4 (i) In your own words, explain what is meant by a 'replicating portfolio'. [3]

(ii) Explain the relevance of risk-neutral probability measures in the pricing of derivatives with payoffs based on the price of an underlying asset. [2]

(iii) Consider a one-period model of a non-dividend paying-stock, currently priced at S_0 and which may move up or down to give $S_1 = S_0 u$ or $S_1 = S_0 d$. Consider a derivative that pays c_u or c_d following an up or down event. The risk-free rate of return (continuously compounded) is r.

(a) Use a replicating portfolio to derive an equation for the price of the derivative at time $t = 0$.

(b) Hence find the price of a derivative whose payoff is defined as $\left| S_1 - S_0 \right|$, assuming $d < 1$ and $u > 1$.

(c) Explain how to synthesise the derivative in (iii)(b) from simpler options. [9]

[Total 14]

X3.5 Using the Black-Scholes formula for the value of a European call option on a non-dividend-paying stock, show that the call price, c, tends to the maximum of $S - Ke^{-r(T-t)}$ and zero (depending on the strike price) as σ tends to zero. [11]

X3.6 The price of a non-dividend-paying stock at time 1, S_1, is related to the price at time 0, S_0, as follows:

$$S_1 = \begin{cases} S_0 u & \text{with probability } p \\ S_0 d & \text{with probability } 1-p \end{cases}$$

The continuously compounded rate of return on a risk-free asset is r.

(i) (a) Determine the replicating portfolio for a European call option written on the stock that expires at time 1 and has a strike price of k, where $dS_0 < k < uS_0$. You should give expressions for the number of units for each constituent in the portfolio.

(b) Use your expressions in (i)(a) to find a formula for the price of the European call option.

(c) Use put-call parity to derive a formula for the price of the corresponding European put option, with the same strike price and expiry date.

(d) Show that the price of the European call option in (i)(b) can be written as the discounted expected payoff under a probability measure Q. Hence find an expression for the probability, q, of an upward move in the stock price under Q.

[11]

(ii) Explain the relationship between the probability measure Q in (i)(d) and the real-world probability measure P, and the relationship you would expect q and p to have if all investors are risk-averse. [3]

[Total 14]

X3.7 (i) Using the put-call parity relationship and the Black-Scholes formula for the value of a European call option on a non-dividend-paying share, c_t, derive the Black-Scholes formula for the value of a European put option on a non-dividend-paying share, p_t, in terms of K, r, $T-t$, d_1 and d_2. [2]

(ii) A non-dividend-paying share has a current value of £20 and an annual volatility of 0.3. An investor who has £100 to invest has a choice between investing in either a one-year zero-coupon bond (redeemable at par) with a current market value of £94.18 or in one-year put options with a strike price of £17.50. If the investor chooses to allocate all of their money to the options, how many can they buy?

[Ignore tax and investment expenses and assume that the bond market and the options market are both arbitrage-free. Assume that the option price is quoted to the nearest penny.] [7]

[Total 9]

X3.8 The price S_t of a particular share follows a geometric random walk:

$$S_t = S_{t-1} Z_t$$

where $\{Z_t\}$ is a sequence of independent, identically distributed random variables:

$$Z_t = \begin{cases} 1.1 & \text{with probability } 0.6 \\ 0.95 & \text{with probability } 0.4 \end{cases}$$

and t denotes the time in months.

A 1-month European call option is available on the share with a strike price of £10.50. The current market price of the share is £10. No dividends are to be paid over the next 6 months. An annualised risk-free force of interest of 4% is available.

(i) Find the expected payoff of the call option. [1]

(ii) Construct a replicating portfolio for the derivative consisting of shares and cash, and hence find the fair price of the derivative. [3]

(iii) Hence state how many options need to be bought (or sold) per share, in order to construct a risk-free portfolio consisting of shares and these options. [1]

(iv) Describe quantitatively the arbitrage opportunity that would arise if the price of the option in the market was equal to the discounted value of the expected payoff. [5]

[Total 10]

X3.9 An investment bank has developed a new exotic derivative, which will pay an amount equal to the share price at maturity multiplied by the share price at maturity less one dollar. Let T be the maturity date of the derivative and r the risk-free force of interest and assume that the Black-Scholes analysis applies.

(i) Use risk-neutral valuation to derive the pricing formula for this derivative at time $t < T$, based on a share that pays no dividends. [9]

(ii) (a) Derive the formula for the delta of the derivative.

(b) Derive a condition for the range of values for the current share price for which delta is positive and comment on what your answer suggests for derivatives of this type with differing terms.

(c) Derive the corresponding formula for the gamma of the derivative and comment on the sign of gamma. [4]

[Total 13]

X4.1 An analyst is using the Merton model, together with the following information, to value the five-year zero-coupon bonds (ZCBs) issued by a company:

- The nominal value of ZCBs issued is \$100 million.
- The company's shares have a market capitalisation of \$118.46 million.
- The volatility of the company's underlying assets has been estimated to be 25% *pa.*
- The five-year risk-free force of interest is 5% *pa*.

(i) Calculate the price per \$100 nominal of a five-year risk-free ZCB. [1]

(ii) Using your answer to (i) to obtain an initial estimate, and then applying linear interpolation, estimate the value of the company's assets (to the nearest \$10,000) and hence show that the value of its ZCBs is \$76.47 million. [5]

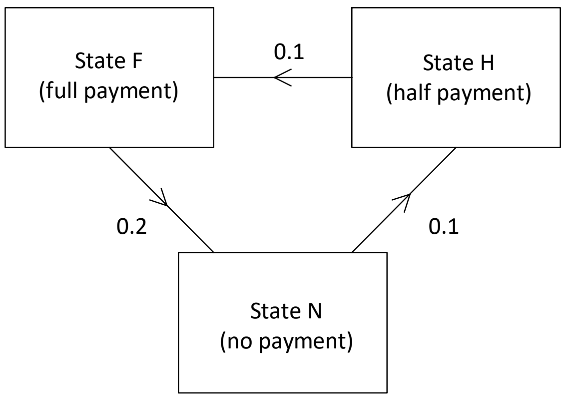
(iii) (a) Use the delta of a European call option based on the Black-Scholes formula (assuming no dividends) to derive a formula for the delta of the ZCBs with respect to the value of the company's assets.

(b) Estimate the numerical value of delta using your calculations in part (ii) and use it to estimate the new value of the ZCBs following a \$10 million fall in the value of the company's assets.

(c) The actual value of the ZCBs following a \$10 million fall in the value of the company's assets is \$76.16 million. Give a possible reason for the discrepancy between your estimated value of the ZCBs and the actual value. [6]

[Total 12]

X4.2 A bank is using a three-state discrete-time Markov chain model to value its bond portfolio.



On 1 January each year the bank assigns each of its client companies to one of the following categories:

- State F: The bank expects to receive any payments due that year in full.
- State H: The bank expects to receive only 50% of any payments due that year.
- State N: The bank expects to receive no payments from the company that year.

The diagram shows the risk-neutral probabilities that each company will move from its current rating level to another level at the time of each review. These probabilities are independent of the company's previous ratings and the behaviour of other companies.

Let $p_{ij}(0,t)$ denote the probability that a company initially in State i will be in State j t years later.

(i) Calculate $p_{Fj}(0,t)$ for $j=F,H$ and $t=1,2,3$. [3]

The bank is considering purchasing at par a 3-year bond issued by a company currently rated as F. Under the terms of the bond, interest of 10% of the face value of the bond will be paid at the end of each year, and the bond will be redeemed at par at the end of the 3 years.

The annual effective yields on 1-year, 2-year and 3-year government bonds are all 5%.

(ii) (a) Calculate the risk-neutral expected present value of the payments from the bond per £100 face value.

(b) Comment on your answer in (ii)(a). [4]

After negotiations, the bank agrees to purchase the bonds at a price of £95.20.

(iii) Calculate the credit spread for this bond. [3]
[Total 10]

X4.3 In the Vasicek model, the spot rate of interest is governed by the stochastic differential equation:

$$dr_t = a\left(b - r_t\right)dt + \sigma\, dB_t$$

where B_t is a standard Brownian motion and a, $b > 0$ are constants.

(i) A stochastic process $\{U_t : t \geq 0\}$ is defined by $U_t = e^{at} r_t$.

(a) Derive an equation for dU_t.

(b) Hence solve the equation to find U_t.

(c) Hence show that:

$$r_t = b + \left(r_0 - b\right)e^{-at} + \sigma \int_0^t e^{a(s-t)} dB_s$$ [5]

(ii) Determine the probability distribution of r_t and the limiting distribution for large t. [4]
[Total 9]

X4.4 (i) Describe briefly the Vasicek one-factor model of interest rates and its key statistical properties. [4]

(ii) According to a particular parameterisation of this model, the instantaneous forward rate applicable at a fixed time $T > t$ implied by the market prices at time t is found to be:

$$f(t,T) = r_t e^{-\alpha\tau} + r_\infty(1-e^{-\alpha\tau}) + k(1-e^{-\alpha\tau})e^{-\alpha\tau}$$

where $\tau = T - t$ and $\alpha > 0$.

Show that, if a humped curve is required for $f(t,T)$, the parameter values must satisfy the condition $k > \left| r_\infty - r_t \right|$. [7]

Here a 'humped curve' means one where the value of the function for some intermediate values of τ exceeds the values for both $\tau = 0$ and $\tau = \infty$. In other words, there will be a maximum value for some positive value of τ.

(iii) In your own words, describe the main advantages and limitations of the Vasicek model. [4]
[Total 15]

X4.5 Claims occur according to a compound Poisson process at a rate of 0.25 claims per year. Individual claim amounts, X, have probability function:

$$P(X = 50) = 0.8$$

$$P(X = 100) = 0.2$$

The insurer charges a premium at the beginning of each year that incorporates a 20% loading factor. The insurer's surplus at time t is $U(t)$. Find $P[U(2) < 0]$ if the insurer starts at time 0 with a surplus of 100. [4]

X4.6 Claims arrive in a Poisson process at rate λ, and $N(t)$ is the number of claims arriving by time t. The claim sizes are independent random variables $X_1, X_2, \ldots$ with mean μ, independent of the arrivals process. The initial surplus is u and the premium loading factor is θ.

(i) (a) Give an expression for the surplus $U(t)$ at time t.

(b) Define the probability of ruin with initial surplus u, $\Psi(u)$.

(c) State the value of $\Psi(u)$ when $\theta = 0$. [3]

(ii) The unit of currency is changed so that one unit of the old currency is worth the same as 2.5 units of the new currency.

Determine a relationship between $\Psi(u)$ in (i)(b) and the probability of ruin for the new process. [2]
[Total 5]

X4.7 Claims arrive in a Poisson process at rate λ. Individual claim amounts are all exactly 100. The insurer applies a premium loading factor of 20%.

(i) (a) Show that the adjustment coefficient, R, satisfies:

$$e^{100R} - 120R - 1 = 0$$

(b) By approximating e^{100R} with a series expansion up to terms in R^3, obtain an approximate value of R. [4]

(ii) Determine the minimum initial capital such that the probability of ruin is at most 0.05. [3]

[Total 7]

X4.8 The table below shows the payments made in each development year in respect of an insurer's claims for fire damage for the three most recent calendar years. You may assume that all claims are paid in the middle of each year.

Claim payments made during year (£'000)		Development year		
		0	1	2
Accident year	2010	830	940	150
	2011	850	920	
	2012	1,120		

The rate of claims inflation over these years, measured over the 12 months to the middle of each year are given below:

Annual claim inflation rate (past)	
2011	2%
2012	2.5%

Estimated annual claim inflation rate (future)	
2013	3%
2014	3%

Use the inflation-adjusted chain ladder method to estimate the total amount outstanding for future claims arising from accident years 2011 and 2012. [8]

X4.9 Cumulative claims incurred on a motor insurance account are as follows:

Cumulative claims incurred (£'000)		Development year		
		0	1	2
Policy year	2010	1,417	1,923	2,101
	2011	1,701	2,140	
	2012	1,582		

The data have already been adjusted for inflation. Annual premiums written in 2012 were £3,073,000 and the ultimate loss ratio has been estimated as 92%. Claims paid to date for policy year 2012 are £441,000, and claims are assumed to be fully run-off by the end of Development year 2.

Estimate the outstanding claims to be paid arising from policies written in 2012 **only**, using the Bornhuetter-Ferguson technique. [7]

X4.10 The following table gives the cumulative incurred claims data, by year of accident and reporting development for a portfolio of motor insurance policies:

Cumulative incurred claims (£'000)		Development year		
		0	1	2
Accident year	2010	252	375	438
	2011	230	343	
	2012	208		

Number of reported claims		Development year		
		0	1	2
Accident year	2010	56	74	87
	2011	49	65	
	2012	44		

(i) Given that the total claims paid to date are £950,000 for Accident years 2010 to 2012 calculate the outstanding claims reserve for this cohort using the average cost per claim method with grossing-up factors. [8]

(ii) State the assumptions that underlie your result. [2]

[Total 10]

X4.11 Aggregate claims on a general insurance company's portfolio form a compound Poisson process with parameter λ.

Individual claims have an exponential distribution with mean 100. The company applies a 20% premium loading. The insurer effects proportional reinsurance with a retained proportion of α. The reinsurer applies a 30% premium loading.

(i) Calculate the minimum value of α such that the insurer's net income is greater than the expected net claims. [3]

(ii) Hence, show that the direct insurer's adjustment coefficient, R, satisfies:

$$R = \frac{1-3\alpha}{100\alpha - 1{,}300\alpha^2}$$ [5]

(iii) By differentiating the result from (ii), show that $\alpha = 0.6257$ maximises the adjustment coefficient and calculate the corresponding optimal value of R.
You may assume that the turning point is a maximum. [5]

[Total 13]