- Define the n-year discrete spot rate of interest.

- What is the discount factor from time n to time 0 using the discrete n-year spot rate of interest?

Discrete spot rate of interest

- The n-year **discrete spot rate of interest**, y_n, is the yield on a unit zero-coupon bond with term n years (*ie* the average annual interest rate over the next n years, starting now).

- The discount factor from time n to time 0 (now) is:

$$\frac{1}{(1+y_n)^n}$$

- What is the definition of a discrete forward rate of interest?

- What is the discount factor from time $t+r$ to time t using the discrete forward rate of interest?

Discrete forward rate of interest

- The r-year **discrete forward rate of interest**, $f_{t,r}$, is the annual interest rate agreed at time 0 for an investment made at time $t > 0$ for a period of r years.

- The discount factor from time $t + r$ to time t is:

$$\frac{1}{(1+f_{t,r})^r}$$

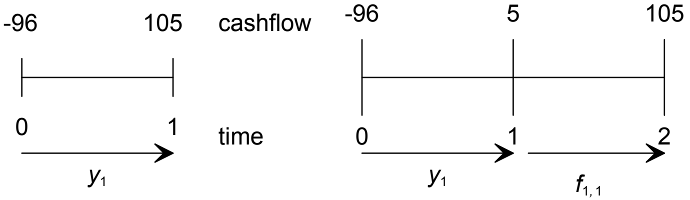
2a

Two bonds paying annual coupons in arrears of 5% and redeemable at par reach their redemption dates in exactly one and two years' time, respectively. The price of each of the bonds is £96 per £100 nominal.

(i) Calculate the 1-year spot rate.

(ii) Calculate the 1-year forward rate that applies from time 1.

Obtaining spot and forward rates from bonds



From the first bond we have: $96 = 105(1+y_1)^{-1} \quad \Rightarrow \quad y_1 = 9.375\%$

From the second bond we have: $96 = 5(1+y_1)^{-1} + 105(1+y_1)^{-1}(1+f_{1,1})^{-1}$

$$\Rightarrow \quad f_{1,1} = 5\%$$

How are continuous-time spot and forward rates related to discrete-time spot and forward rates?

Continuous-time spot and forward rates

The continuous-time spot/forward rate is the force of interest that is equivalent to the annual effective (*ie* discrete-time) spot/forward rate.

So, if Y_t is the *t*-year continuous-time spot rate and $F_{t,r}$ is the *r*-year continuous-time forward rate from time *t*, then:

$$Y_t = \ln(1 + y_t)$$

$$F_{t,r} = \ln(1 + f_{t,r})$$

eg a 3% *pa* discrete-time spot/forward rate is equivalent to a continuous-time spot/forward rate of $\ln(1.03) = 2.9559\%$ *pa.*

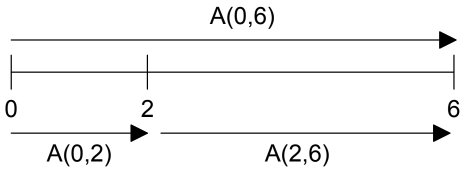
How do we convert between spot and forward rates?

Converting between spot and forward rates

We equate equivalent accumulation (or discount) factors. Considering the time period from time 0 to time *t* and then forward to time $t+r$:

$$(1+y_{t+r})^{t+r} = (1+y_t)^t (1+f_{t,r})^r$$

For example:



$$A(0,6) = A(0,2)A(2,6) \quad \Rightarrow \quad (1+y_6)^6 = (1+y_2)^2 (1+f_{2,4})^4$$

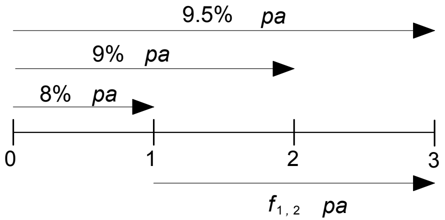
4a

At time 0 the 1-year spot rate is 8% *pa*, the 2-year spot rate is 9% *pa*, and the 3-year spot rate is 9½% *pa* effective.

(i) Calculate the 2-year (discrete-time) forward rate from time 1.

(ii) Calculate the *continuous* 2-year forward rate from time 1.

__Converting between spot and forward rates__



(i) $A(0,1)A(1,3) = A(0,3) \quad \Rightarrow \quad A(1,3) = \frac{A(0,3)}{A(0,1)}$

$$\Rightarrow \quad (1+f_{1,2})^2 = \frac{1.095^3}{1.08} \quad \Rightarrow \quad f_{1,2} = 10.26\%$$

(ii) $F_{1,2} = \ln 1.1026 = 9.77\%$

Define the instantaneous forward rate F_t, and give a formula for it in terms of P_t, the price at time 0 of a zero-coupon bond with redemption payment 1 at time *t*.

Instantaneous forward rates

The instantaneous forward rate F_t is defined as:

$$F_t = \lim_{r \to 0} F_{t,r}$$

and can broadly be thought of as the forward force of interest applying over the instant of time from t to $t+h$, where h is small.

In terms of P_t, the price at time 0 of a zero-coupon bond with redemption payment 1 at time t:

$$F_t = -\frac{1}{P_t}\frac{d}{dt}P_t$$

What are the names of the three most popular theories that explain why interest rates vary with term?

Theories that explain why interest rates vary with term

- Expectations theory
- Liquidity preference theory
- Market segmentation theory

Describe the expectations theory.

Expectations theory

The relative attraction of short and longer-term investments will vary according to expectations of future movements in interest rates.

An expectation of a fall in interest rates will make short-term investments less attractive and longer-term investments more attractive.

In these circumstances yields on short-term investments will rise and yields on long-term investments will fall.

An expectation of a rise in interest rates will have the converse effect.

Describe the liquidity preference theory.

Liquidity preference theory

Long-dated bonds are more sensitive to interest-rate movements than short-dated bonds.

Risk-averse investors will require compensation (in the form of higher yields) for the greater risk of loss on longer bonds.

This theory would predict a higher return on long-term bonds than short-term bonds.

9

Describe the market segmentation theory.

Market segmentation theory

Bonds of different terms attract different investors, (who will choose assets that are similar in term to their liabilities).

For example, banks will invest in short-term bonds (since their liabilities are very short term) and pension funds will invest in the longest-dated bonds (since their liabilities are long-term).

Therefore, the demand for bonds will differ for different terms.

The supply of bonds will also vary by term, as governments’ and companies’ strategies may not correspond to investors’ requirements.

These different forces of supply and demand will lead to the term structure of interest rates.

Define the gross redemption yield.

Gross redemption yield

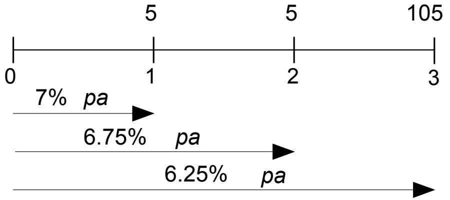
The **gross redemption yield** is the effective interest rate that satisfies the equation of value for a bond (where there is no allowance for tax).

10a

At time 0, the 1-year spot rate is 7% *pa*, the 2-year spot rate is 6¾% *pa* and the 3-year spot rate is 6¼% *pa*. Calculate the price of a 3-year bond, redeemable at par with annual coupons of 5% *pa* payable in arrears.

Price of bond

At time 0, the 1-year spot rate is 7% *pa*, the 2-year spot rate is 6¾% *pa* and the 3-year spot rate is 6¼% *pa*. A 3-year bond is redeemable at par with annual coupons of 5% *pa* payable in arrears.



The price of the bond is:

$$P = \frac{5}{1.07} + \frac{5}{1.0675^2} + \frac{105}{1.0625^3} = 96.60$$

10b

Calculate the gross redemption yield of a 3-year bond, redeemable at par with annual coupons of 5% *pa* payable in arrears. The price of the bond is 96.60.

Gross redemption yield

The price of a 3-year bond, redeemable at par with annual coupons of 5% *pa* payable in arrears is 96.60.

The gross redemption yield is the effective interest rate that solves the equation of value:

$$96.60 = 5a_{\overline{3}|} + 100v^3$$

At 6.2%, the right-hand side is 96.804.
At 6.3%, the right-hand side is 96.544.

Interpolating, we get:

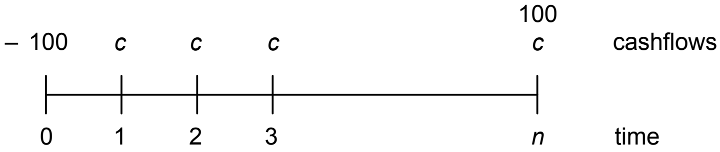
$$\frac{i - 6.2}{6.3 - 6.2} = \frac{96.60 - 96.804}{96.544 - 96.804} \quad \Rightarrow \quad i = 6.28\%$$

- Define the *n*-year par yield.

- Define the coupon bias.

Par yield and coupon bias

- The n-year **par yield** is the annual coupon rate, c, such that the price paid for an n-year bond paying coupons annually in arrears and redeemed at par is 100 per 100 nominal.



- The **coupon bias** is the difference between the n-year par yield and the n-year spot rate.

11a

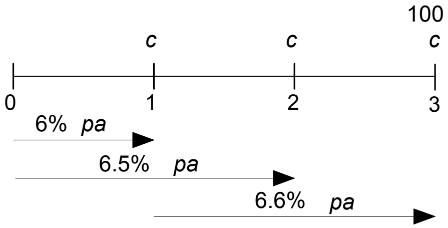
The n-year forward rate for transactions beginning at time t and maturing at time $t+n$ is denoted as $f_{t,n}$. You are given:

$$f_{0,1} = 6.0\% \text{ per annum}$$
$$f_{0,2} = 6.5\% \text{ per annum}$$
$$f_{1,2} = 6.6\% \text{ per annum}$$

Calculate the 3-year par yield.

Par yield

We are given that $f_{0,1} = 6\%\,pa$, $f_{0,2} = 6.5\%\,pa$ and $f_{1,2} = 6.6\%\,pa$.



The par yield can be found from the equation:

$$100 = \frac{c}{1.06} + \frac{c}{1.065^2} + \frac{100 + c}{1.06 \times 1.066^2} \quad \Rightarrow \quad c = 6.40$$

So the par yield is 6.4%.

- Define the discounted mean term.
- What is another name for the discounted mean term?
- State the formula for modified duration in terms of the discounted mean term.

Discounted mean term

The **discounted mean term** is the average time of the cashflows, weighted by present value:

$$\tau = DMT = \frac{\sum t \times PV}{\sum PV} = \frac{\sum tcv^t}{\sum cv^t}$$

Another name for the discounted mean term is the **Macaulay duration.**

The **modified duration** is defined as:

$$\frac{DMT}{1+\frac{i^{(p)}}{p}}$$

12a

Calculate the discounted mean term, at an effective annual interest rate of 8%, for a 5-year fixed-interest security with coupons of 10% *pa* paid half-yearly in arrears, redeemable at par.

Discounted mean term

We will work in half-years, using an effective rate of interest of:

$$1.08^{\frac{1}{2}} - 1 = 3.923\%$$

$$PV = 5a_{\overline{10|}} + 100v^{10} = 108.77$$

$$DMT = \frac{1 \times 5v + 2 \times 5v^2 + \cdots + 10 \times 5v^{10} + 10 \times 100v^{10}}{PV}$$

$$= \frac{5(Ia)_{\overline{10|}} + 1{,}000v^{10}}{PV} = \frac{5 \times 42.203 + 1{,}000 \times 1.03923^{-10}}{PV} = 8.2$$

So the discounted mean term is 4.1 years.

13

- Define volatility.
- Give another name for volatility.
- State the relationship between the discounted mean term and the volatility.

Volatility

The **volatility** is defined to be:

$$vol(i) = -\frac{PV'(i)}{PV(i)} = -\frac{\frac{d}{di}PV(i)}{PV(i)}$$

where PV is the present value of the cashflows.

Another name for volatility is **effective duration**.

The **discounted mean term** and the **volatility** are connected by the equation:

$$DMT = (1+i) \times vol(i) \quad \text{or, equivalently} \quad vol(i) = v \times DMT$$

13a

A loan stock is issued that pays coupons annually in arrears at 8% *pa* and is redeemable at par after 10 years.

Calculate the volatility of this stock on the date of issue at an effective rate of interest of 8% *pa*.

Volatility

A loan stock is issued that pays coupons annually in arrears at 8% *pa* and is redeemable at par after 10 years. If $i = 8\%$ *pa*:

$$P(i) = 8a_{\overline{10}|} + 100v^{10} = 8\sum_{t=1}^{10}(1+i)^{-t} + 100(1+i)^{-10}$$

$$P(0.08) = 100$$

$$P'(i) = -8\sum_{t=1}^{10} t(1+i)^{-(t+1)} - 1{,}000(1+i)^{-11} = -8v\,(Ia)_{\overline{10}|} - 1{,}000v^{11}$$

$$P'(0.08) = -671.0$$

So the volatility is $-\dfrac{P'(0.08)}{P(0.08)} = 6.71$.

14

Consider a series of n cashflows, where C_k is the cashflow at time t_k ($k = 1,2,...,n$). Prove that $vol(i) = v \times DMT$.

Proof that $\boldsymbol{vol(i) = v \times DMT}$

For this series of cashflows:

$$PV(i) = \sum_{k=1}^{n} C_k v^{t_k} = \sum_{k=1}^{n} C_k (1+i)^{-t_k}$$

and: $$PV'(i) = \sum_{k=1}^{n} (-t_k) C_k (1+i)^{-(t_k+1)}$$

So:

$$vol(i) = -\frac{\sum_{k=1}^{n} (-t_k) C_k (1+i)^{-(t_k+1)}}{\sum_{k=1}^{n} C_k (1+i)^{-t_k}} = \frac{v \times \sum_{k=1}^{n} t_k C_k v^{t_k}}{\sum_{k=1}^{n} C_k v^{t_k}} = v \times DMT$$

Define convexity.

Convexity

The **convexity** of a cashflow series is defined to be:

$$conv(i) = \frac{PV''(i)}{PV(i)} = \frac{\frac{d^2}{di^2} PV(i)}{PV(i)}$$

Note that convexity is a measure representing the spread of payments around the discounted mean term. Typically, the more spread out the cashflows, the higher the convexity.

15a

An investor has a liability of £100,000 to be paid in 7.247 years' time.

Calculate the volatility and convexity of this liability at an effective rate of interest of 8% *pa.*

Volatility and convexity

$P(i) = 100{,}000v^{7.247}$ $\qquad$ $P(0.08) = 57{,}250.34$

$P'(i) = -7.247 \times 100{,}000v^{8.247}$ $\qquad$ $P'(0.08) = -384{,}160.36$

$P''(i) = 8.247 \times 7.247 \times 100{,}000v^{9.247}$ $\qquad$ $P''(0.08) = 2{,}933{,}491.19$

So the volatility is: $\dfrac{384{,}160.36}{57{,}250.34} = 6.71$

and the convexity is: $\dfrac{2{,}933{,}491.19}{57{,}250.34} = 51.24$

State Redington's conditions for immunisation.

Redington's conditions for immunisation

- $PV_{assets} = PV_{liabs}$

- $vol_{assets} = vol_{liabs}$ or $DMT_{assets} = DMT_{liabs}$

  or $PV'_{assets} = PV'_{liabs}$

- $conv_{assets} > conv_{liabs}$ or $PV''_{assets} > PV''_{liabs}$

These conditions must hold at the current interest rate.

If a fund is immunised then small changes in the interest rate will result in a profit being made on the fund.

16a

A loan stock issued on 1 March 2018 has coupons payable annually in arrears at 8% *pa.* Capital is to be redeemed at par 10 years from the date of issue. The volatility of this stock at 1 March 2018 at an effective rate of interest of 8% *pa* is 6.71. At 1 March 2018 an investor has a liability of £100,000 to be paid in 7.247 years' time, which also has a volatility of 6.71 and convexity of 51.24.

On 1 March 2018 the investor decides to invest a sum equal to the present value of the liability in the loan stock, where the present value of the liability and the price of the loan stock are both calculated at an effective rate of interest of 8% *pa.* Given that the convexity of the loan stock at 1 March 2018 is 60.53, state with reasons whether the investor will be immunised against small movements in interest rates on that date.

Immunisation

The present value of the assets is equal to the present value of the liabilities and the volatility of the assets is equal to the volatility of the liabilities, so conditions 1 and 2 are satisfied.

Since the convexity of the assets is 60.53 and the convexity of the liabilities is 51.24, the third condition is also satisfied and immunisation has been achieved.

16b

An investor has to pay a lump sum of £20,000 in 15 years' time. The investor wishes to immunise these liabilities by investing in two zero-coupon bonds, Bond X and Bond Y. The effective rate of interest is 7% per annum. The investor has decided to invest an amount in Bond X sufficient to provide a capital sum of £10,000 when Bond X is redeemed in ten years' time.

Determine the term needed for Bond Y.

Immunisation – term for Bond Y

If Y is the amount invested in Bond Y, then equating present values:

$$PV_L = PV_A \Rightarrow 20{,}000v^{15} = 10{,}000v^{10} + Y$$

$$\Rightarrow Y = 2{,}165.43$$

The DMT of the assets must equal the DMT of the liabilities and so if n is the term of Bond Y then:

$$\frac{10X + nY}{X + Y} = 15 \Rightarrow n = \frac{15 \times 20{,}000v^{15} - 10 \times 10{,}000v^{10}}{2{,}165.43}$$

$$\Rightarrow n = 26.7 \text{ years}$$

The spread of the assets around the DMT, and hence the convexity, is greater than that of the liabilities and so immunisation is achieved.

List the practical problems faced when trying to immunise a fund.

Practical problems with immunising a fund

- **Requires continuous rebalancing** of the asset portfolio to keep the asset and liability PV/volatilities equal. This will take time and incur dealing costs.
- There may be **options or uncertainties** in the asset or liability cashflows, meaning that the cashflows are estimated rather than being known with certainty.
- **Assets may not exist** to provide the necessary overall asset volatility to match the liability volatility.
- Redington's theory only provides immunisation against **small changes** in the interest rate.
- Redington's theory assumes a **flat yield curve** (*ie* the interest rate is the same at all durations) and that the interest rate changes by the same amount at all durations, which may not be the case in reality.

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