FAC

Corrections to 2014 study material

Comment

This document contains details of any errors and ambiguities in the FAC study materials for 2014 that have been brought to our attention. We will incorporate these changes in the study material each year. We are always happy to receive feedback from students, particularly details concerning any errors, contradictions or unclear statements in the courses. If you have any such comments on this course please email them to FAC@bpp.com.

Important note

This document was last revised significantly on 3rd June 2014. The date on which any corrections have been added is noted at the start of each section.

Chapter 4 (updated on 30th January 2014)

Page 37

There is a typo on the range of values for which the series converges. It should read:

$$-1 < x < 1$$

Replacement pages can be found at the end.

Chapter 5 (updated on 3rd June 2014)

Page 30

The percentage errors have the numerators the wrong way round. They should read:

$$\frac{1.027^{-1} - 1.0265^{-1}}{1.0265^{-1}} \times 100\% = -0.048685\%$$

and:

$$\frac{1.027^{-1} - 1.0275^{-1}}{1.0275^{-1}} \times 100\% = 0.048685\%$$

Replacement pages can be found at the end.

Chapter 6 (updated on 13th November 2013)

Summary Page 34

There is a typo on the Stationary (turning) points section. The second condition should read:

$$\frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad \text{min}$$

Replacement pages can be found at the end.

Chapter 7 (updated on 3rd June 2014)

Page 49

There is a typo in the second paragraph. It should read:

This will give an over estimate of the true value since the gradient of the curve is **decreasing** over the range of values.

Replacement pages can be found at the end.

Q&A Bank

Question A63 Page 25 (updated on 3rd June 2014)

The question asks for a proportionate change but the answers are given as percentages. The answers should read:

A 0.1429
B 0.1667
C 0.8333
D 1.167

Replacement pages can be found at the end.

Solution A54 Page 17 (updated on 30th January 2014)

The answer C is correct but the solution should read:

$$\begin{aligned}\sum_{x=1}^{6}\sum_{y=x}^{6}2x(y-1) &= \sum_{y=1}^{6}2(y-1)\sum_{x=1}^{y}x \\ &= \sum_{y=1}^{6}2(y-1)\tfrac{1}{2}y(y+1) \\ &= \sum_{y=1}^{6}y^3-y \\ &= \sum_{y=1}^{6}y^3-\sum_{y=1}^{6}y \\ &= \tfrac{1}{4}6^2(6+1)^2-\tfrac{1}{2}6(6+1) \\ &= 420\end{aligned}$$

Replacement pages can be found at the end.

Solution A63 Page 22 (updated on 3rd June 2014)

The question asks for a proportionate change but the solution calculates a percentage change. It should read:

The change is $\frac{30{,}000}{180{,}000} = 0.1667$.

Replacement pages can be found at the end.

9.2 Fractional or negative powers

Very often in practice, the power on the bracket will not be a positive integer, so alternatively we can use:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

where n is negative or fractional, and $-1 < x < 1$.

This series is derived using Maclaurin series which is covered in Chapter 7.

Example

Expand $\sqrt{(1-2x)}$ as far as the term in x^3. For what values of x is this expansion valid?

Solution

$$\sqrt{(1-2x)} = (1-2x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(-2x) + \frac{\tfrac{1}{2}(-\tfrac{1}{2})}{2!}(-2x)^2 + \frac{\tfrac{1}{2}(-\tfrac{1}{2})(-\tfrac{3}{2})}{3!}(-2x)^3 + \cdots$$

$$= 1 - x - \tfrac{1}{2}x^2 - \tfrac{1}{2}x^3 + \cdots$$

This expansion is valid for $-1 < -2x \le 1$, or $-\tfrac{1}{2} \le x < \tfrac{1}{2}$.

Question 4.23

Expand $(1+x)^{\frac{1}{2}}$ and hence approximate the following (without using a calculator!):

$$1.04^{\frac{1}{2}},\ 1.10^{\frac{1}{4}},\ 0.99^{-\frac{1}{2}},\ \frac{1.09^{\frac{1}{2}}}{1.07^{\frac{1}{2}}},\ \sqrt{10200},\ \frac{a^{\frac{1}{2}} - b^{\frac{1}{2}}}{a-b}$$

where $a = 1.11$ and $b = 1.10$.

Example

Expand $\dfrac{1+x}{2+3x}$ as far as the term in x^3. For what values of x is this expansion valid?

Solution

We start by manipulating the expression, since we want a binomial expansion which starts with a 1:

$$\frac{1+x}{2+3x}=(1+x)(2+3x)^{-1}=2^{-1}(1+x)(1+\tfrac{3}{2}x)^{-1}$$

$$=2^{-1}(1+x)\left(1-\tfrac{3}{2}x+\frac{(-1)(-2)}{2!}\left(\tfrac{3}{2}x\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\tfrac{3}{2}x\right)^3+\cdots\right)$$

$$=2^{-1}(1+x)\left(1-\tfrac{3}{2}x+\tfrac{9}{4}x^2-\tfrac{27}{8}x^3+\cdots\right)$$

$$=\tfrac{1}{2}\left(1-\tfrac{3}{2}x+\tfrac{9}{4}x^2-\tfrac{27}{8}x^3+x-\tfrac{3}{2}x^2+\tfrac{9}{4}x^3+\cdots\right)$$

$$=\tfrac{1}{2}-\tfrac{1}{4}x+\tfrac{3}{8}x^2-\tfrac{9}{16}x^3+\cdots$$

This is valid for $-1<\tfrac{3}{2}x\le 1$, *ie* $-\tfrac{2}{3}<x\le\tfrac{2}{3}$.

Question 4.24

Expand $\dfrac{1}{\sqrt{(1-4x)}}$ as far as the term in x^3. For what values of x is this expansion valid? Use your expansion to estimate $0.98^{-\frac{1}{2}}$ to 6DP.

Question 4.25

Simplify $S_1=\sum_{x=0}^{\infty}\binom{k+x-1}{x}p^k q^x$, and $S_2=\sum_{x=0}^{n}x\binom{n}{x}p^x q^{n-x}$, given that $p+q=1$ and k is a positive integer.

Chapter 5 Solutions

Solution 5.1

(i) In each year the population increases by 5%, so the new population is $100\% + 5\% = 105\%$ of the old population. So we will multiply by 1.05 each year. If the starting population is x then we need to find out how many years we multiply by 1.05 before we get $2x$:

$$x \times 1.05^n = 2x \quad \Rightarrow \quad 1.05^n = 2$$

Taking logs of both sides and rearranging:

$$n \ln 1.05 = \ln 2 \quad \Rightarrow \quad n = \frac{\ln 2}{\ln 1.05} = 14.2067$$

So we will need at least 15 years for the population to double in size.

(ii) Each year we will multiply by 1.05. So using x to stand for the population in 2010 then we have:

$$x \times 1.05^2 = 66{,}150{,}000 \quad \Rightarrow \quad x = \frac{66{,}150{,}000}{1.05^2} = 60{,}000{,}000$$

Solution 5.2

0.66% is equivalent to saying 6.6‰.

Solution 5.3

Absolute change is £467.

Proportionate change is 0.0321.

Percentage change is 3.21%.

Solution 5.4

The true value could have been any number from 0.0265 to 0.0275. So we will use the extreme precise values to find the largest percentage error.

If $i = 0.0265$ then the true calculation would have been $v = 1.0265^{-1}$ compared to the approximate calculation of $v = 1.027^{-1}$ so this gives a percentage error of:

$$\frac{1.027^{-1} - 1.0265^{-1}}{1.0265^{-1}} \times 100\% = -0.048685\%$$

Similarly, if $i = 0.0275$ then the percentage error would be:

$$\frac{1.027^{-1} - 1.0275^{-1}}{1.0275^{-1}} \times 100\% = 0.048685\%$$

In this case they are both the same magnitude to 5 significant figures. So the largest percentage change is 0.048685% (whether up or down).

Solution 5.5

Since the variance is calculated by squaring the x values, the dimension of the variance here is T^2, where T is the dimension of time.

Solution 5.6

The other possible formulae are (i) and (iii). The second one can't be correct because the dimension of the first term is T^2, while that of the second term is T, and terms of different dimensions cannot be subtracted.

Solution 5.7

Consider the series for e^x which is $1 + x + \frac{x^2}{2!} + \cdots$. If x had a dimension, then each term of this series would have a different dimension and so this series would be meaningless. Therefore x must be dimensionless.

A similar reasoning applies to $\log x$.

Chapter 6 Summary

Order notation

$f(x)$ is $O(g(x))$ if there exists at least one constant $K > 0$ such that:

$$\left|\frac{f(x)}{g(x)}\right| \le K \text{ for all } x < x_0$$

$f(x)$ is $o(g(x))$ if the inequality is true for all constants $K > 0$, which means:

$$\lim_{x\to 0}\left|\frac{f(x)}{g(x)}\right| = 0$$

Differentiation

Differentiating a function gives the gradient of the function at a general point. It is defined as:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

Standard results include:

Function	**Derivative**
x^n	$nx^{n-1} \quad n \ne 0$
c^x	$c^x \ln c$
e^x	e^x
$\ln x$	$\frac{1}{x}$

Product and Quotient rules

If u and v are functions of x, then:

$$\frac{d}{dx}(uv) = u'v + uv'$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

The chain rule:

To differentiate nested functions, we use:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This gives the following shortcut rules:

Function	***Derivative***
$[f(x)]^n$	$n[f(x)]^{n-1} \times f'(x)$
$e^{f(x)}$	$e^{f(x)} \times f'(x)$
$\ln f(x)$	$\frac{1}{f(x)} \times f'(x)$

Stationary (turning) points

These occur when $\frac{dy}{dx} = 0$. To determine their nature, we use:

$$\frac{d^2y}{dx^2} < 0 \quad \Rightarrow \quad \text{max} \qquad\qquad \frac{d^2y}{dx^2} > 0 \quad \Rightarrow \quad \text{min}$$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \text{point of inflexion (but needs further investigation)}$$

In some situations to find the maximum or minimum of a function it is easier to find the maximum or minimum of the log of the function.

Curve sketching

- Find where the function crosses the x and y axes
- Find any stationary points and their nature
- Consider the sign and gradient of the function
- Consider the behaviour of the function at extreme values or at impossible values.

Then integrating with respect to y, we get:

$$\int_{x=0}^{30} e^{-\nu x} e^{-\delta x} \left[-e^{-\mu y} \right]_x^{30} dx$$

$$= \int_{x=0}^{30} e^{-\nu x} e^{-\delta x} \left[e^{-\mu x} - e^{-30\mu} \right] dx$$

$$= \int_{x=0}^{30} e^{-(\nu+\delta+\mu)x} - e^{-(\nu+\delta)x} e^{-30\mu} \ dx$$

$$= \left[-\frac{1}{\nu+\delta+\mu} e^{-(\nu+\delta+\mu)x} + \frac{1}{\nu+\delta} e^{-(\nu+\delta)x} e^{-30\mu} \right]_0^{30}$$

$$= \frac{1}{\nu+\delta+\mu} (1 - e^{-30(\nu+\delta+\mu)}) + \frac{1}{\nu+\delta} (e^{-30(\nu+\delta+\mu)} - e^{-30\mu})$$

Substituting in the given values, we also get 1.213, confirming that the two methods give the same numerical value.

Solution 7.18

Using 5 ordinates means that we need to split the area into 4 sections, each of width 0.25. The trapezium rule will then give the approximate area to be:

$$\frac{1}{2}\left(\frac{2}{2} + \frac{4}{2.25} + \frac{4}{2.5} + \frac{4}{2.75} + \frac{2}{3} \right) \times 0.25 = 0.812$$

This will give an over estimate of the true value since the gradient of the curve is decreasing over the range of values.

We can check the exact value by integrating:

$$\int_2^3 \frac{2}{x} \, dx = \left[2\ln|x| \right]_2^3 = 2\ln 3 - 2\ln 2 = 0.811$$

Solution 7.19

$f(i)=(1+i)^{-2}+(1+i)^{-5}$, so $f(0.1)=1.447$, and:

$$f'(i)=-2(1+i)^{-3}-5(1+i)^{-6}$$

$$f''(i)=6(1+i)^{-4}+30(1+i)^{-7}$$

$$f'''(i)=-24(1+i)^{-5}-210(1+i)^{-8}$$

giving:

$$f'(0.1)=-4.325$$

$$f''(0.1)=19.493$$

$$f'''(0.1)=-112.869$$

Using the Taylor series, this gives:

$$(1+i)^{-2}+(1+i)^{-5}=1.447-4.325(i-0.1)+19.493\frac{(i-0.1)^2}{2!}-112.869\frac{(i-0.1)^3}{3!}$$

(as far as the term in $(i-0.1)^3$).

Because the term $-4.325(i-0.1)$ has a negative coefficient, increasing i would reduce the value of the function. In fact, changing i to 0.11 (*ie* $i-0.1=0.01$) would increase the value of the function by approximately $-4.325\times 0.01=-0.04325$.

Chapter 5

Question A61

The population of a country at the start of the year is 62,400,000. The population at the end of the year is 63,760,000. Express the change in population as a rate per mil.

A 2.13‰
B 2.18‰
C 21.8‰
D 21.3‰ [1]

FAC 5 1

Question A62

An item has increased in cost by 30% to $58.50. What was the original cost?

A $38.50
B $40.95
C $45
D $76.05 [1]

FAC 5 1

Question A63

What is the proportionate change in the price of a house that goes from £180,000 to £210,000?

A 0.1429
B 0.1667
C 0.8333
D 1.167 [1]

FAC 5 2

Question A64

When calculating $y = \sqrt[3]{x}$, a student uses $x = 50$ to 2 significant figures. What is the maximum absolute error possible here?

A 0.0122
B 0.0123
C 0.119
D 0.127 [1]

FAC 5 3

Question A65

You know that a equals 5,000 correct to one significant figure and b equals 0.20 correct to two decimal places. You calculate a/b to be 25,000. What is the greatest percentage by which you might be overestimating the true value?

A 8.78%
B 11.4%
C 12.8%
D 13.9% [2]

FAC 5 3

Question A66

The following values from a standard normal distribution table were obtained:

$$P(Z < 1.20) = 0.88493 \qquad P(Z < 1.21) = 0.88686$$

Use linear interpolation to calculate the value of $P(Z < 1.207)$.

A 0.88507
B 0.88551
C 0.88590
D 0.88628 [1]

FAC 5 5

Solution A52

D

$$\begin{aligned}\sum_{i=1}^{n}(1+2i+i^2) &= \sum_{i=1}^{n}1+2\sum_{i=1}^{n}i+\sum_{i=1}^{n}i^2 \\ &= n+2\left(\tfrac{1}{2}n(n+1)\right)+\tfrac{1}{6}n(n+1)(2n+1) \\ &= n+n^2+n+\tfrac{1}{6}(2n^3+3n^2+n) \\ &= \frac{2n^3+9n^2+13n}{6}\end{aligned}$$

Solution A53

D

Considering the order of x and y we have $1 \le y \le x \le 20$. So if x sums over the numbers 1 to 20, then looking at the order y must sum from 1 to x.

Solution A54

C

Swapping the order of summation gives:

$$\begin{aligned}\sum_{x=1}^{6}\sum_{y=x}^{6}2x(y-1) &= \sum_{y=1}^{6}2(y-1)\sum_{x=1}^{y}x \\ &= \sum_{y=1}^{6}2(y-1)\tfrac{1}{2}y(y+1) \\ &= \sum_{y=1}^{6}y^3-y \\ &= \sum_{y=1}^{6}y^3-\sum_{y=1}^{6}y \\ &= \tfrac{1}{4}6^2(6+1)^2-\tfrac{1}{2}6(6+1) \\ &= 420\end{aligned}$$

Solution A55

B

The term will be ${}^8C_5(2x)^5(5y)^3 = 56\times 2^5\times 5^3 x^5y^3 = 224{,}000x^5y^3$

Solution A56

A

$$\begin{aligned}\frac{(2-5x)}{(3x+2)^2} &= \frac{(2-5x)}{2^2(\frac{3}{2}x+1)^2}\\ &= \tfrac{1}{4}(2-5x)(1+\tfrac{3}{2}x)^{-2}\\ &= \tfrac{1}{4}(2-5x)\left[1+(-2)(\tfrac{3}{2}x)+\tfrac{(-2)(-3)}{2!}(\tfrac{3}{2}x)^2+\tfrac{(-2)(-3)(-4)}{3!}(\tfrac{3}{2}x)^3+\cdots\right]\\ &= \tfrac{1}{4}(2-5x)(1-3x+\tfrac{27}{4}x^2-\tfrac{27}{2}x^3+\cdots)\\ &= \tfrac{1}{4}\left[(2-6x+\tfrac{27}{2}x^2-27x^3+\cdots)+(-5x+15x^2-\tfrac{135}{4}x^3+\cdots)\right]\\ &= \tfrac{1}{2}-\tfrac{11}{4}x+\tfrac{57}{8}x^2-\tfrac{243}{16}x^3+\cdots\end{aligned}$$

Solution A57

(i) ***Approximate value for i***

Using the approximate relationship, the equation given becomes:

$$20{,}000(1+i)+5{,}000(1+\tfrac{1}{2}i)+2{,}500(1+\tfrac{1}{4}i)=30{,}000 \qquad [1]$$

ie $27{,}500+23{,}125i=30{,}000 \quad\Rightarrow i=10.81\%$ [2]

We could have divided this through by 100 to make the figures easier.

(ii) I***mproved approximate value for i***

If we included the quadratic term in the approximate relationship, it would become:

$$(1+i)^t \approx 1+ti+\tfrac{1}{2}t(t-1)i^2 \qquad [1]$$

Solution A60

(i) ***Show that***

We have:

$$\frac{S_n(x)}{x} = 1 + 2x + 3x^2 + \cdots nx^{n-1}$$

$$S_n(x) = x + 2x^2 + \cdots + (n-1)x^{n-1} + nx^n$$

Subtracting gives:

$$\frac{S_n(x)}{x} - S_n(x) = 1 + x + x^2 + \cdots + x^{n-1} - nx^n \qquad [1]$$

The LHS can be factorised and the terms except the last on the RHS can be summed as a GP:

$$\left(\frac{1}{x} - 1\right) S_n(x) = \frac{1 - x^n}{1 - x} - nx^n \quad \Rightarrow \quad \left(\frac{1 - x}{x}\right) S_n(x) = \frac{1 - x^n}{1 - x} - nx^n \qquad [1]$$

Multiplying both sides by $\left(\frac{x}{1-x}\right)$, then gives the required result:

$$S_n(x) = \frac{x(1 - x^n)}{(1 - x)^2} - \frac{nx^{n+1}}{1 - x} \qquad [1]$$

Comment

This multiply-and-subtract "trick" can be a useful for summing certain types of series. This method is used for deriving formulae for annuity functions in Subject CT1.

(ii) ***Limit and convergence***

Provided $-1 < x < 1$, x^n will vanish as $n \to \infty$ and we will have:

$$\lim_{n \to \infty} S_n(x) = \frac{x}{(1 - x)^2} \qquad [2]$$

Solution A61

C

The change is $\frac{1,360,000}{62,400,000} = 0.0218 = 21.8‰$

Solution A62

C

$\$58.50 \div 1.3 = \$45$.

Solution A63

B

The change is $\frac{30,000}{180,000} = 0.1667$.

Solution A64

B

50 to 2 SF gives a range of 49.5 to 50.5.

$$\sqrt[3]{50.5} - \sqrt[3]{50} = 3.6963 - 3.6840 = 0.0122$$
$$\sqrt[3]{50} - \sqrt[3]{49.5} = 3.6840 - 3.6717 = 0.0123$$

So the maximum absolute error is 0.0123.